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Partitioning: Construction of Rational Number in Young Children

by

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A THESIS

SUBMITTED TO THE FACULTY OF GRADUATE STUDIES AND RESEARCH
IN PARTIAL FULFILMENT OF THE REQUIREMENTS FOR THE DEGREE
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Date July 24, 1981

DEDICATION

To Dr. D. Sawada and Dr. T. E. Kieren

True Educators

In Appreciation

ABSTRACT

The purpose of the present study was to describe and interpret the partitioning behaviors of young children in order to gain insights into the development of the partitioning mechanism; a mechanism purported to function as a foundational element in rational number development.

Five tasks were devised for the study, each one embodying materials which differed in substantive nature, varied in mathematical difficulty, and admitted a modifiable partitioning procedure.

A clinical interaction technique framed within a discovery paradigm was the methodology employed. The researcher interacted with the respondents in the context of controlled yet flexible clinical sessions; sessions which were characterized by intense observation, questioning, and dynamic involvement. The clinical sessions were videotaped and audiotaped and later transcribed.

Following the method of theoretical sampling and in accord with the purpose of the study, the analysis process and the selection of respondents were ongoing throughout the period of data collection. The analysis process endured through three stages; immersion (during clinical interactions), reflection (between clinical sessions), and documentation (following the data collection period).

The respondents, all acquainted with the researcher, were chosen on the basis that they would freely interact during a clinical session. The sample consisted of 43 children from two kindergartens, a grade one, two, and three class in a public school within the Yellowhead School Division, Alberta.

One task, the Cake Problem, proved to be highly effective in enabling children to demonstrate partitioning capabilities and techniques; therefore, the descriptive data and the major findings relate specifically to this problem.

From insights gained during the data analysis, a theory concerning the development of the partitioning mechanism has been formulated and is presented in the form of two propositions.

Proposition I identifies conceptual structures which undergird the partitioning mechanism; structures which have their genesis in basic number theory and transformation geometry concepts. Proposition II presents a hierarchy of partitioning capabilities in five stages.

The proposed stages are presented here in point form while highlighting three distinctive characteristics, namely: the operator or key concept developing during the stage, the algorithm or procedure employed to produce the partitions, and the domain or the extent of the partitioning capabilities within the stage.

The stages are:

Stage I: The Sharing Stage

Operator: - breaking; sharing; half it.

Algorithm: - allocating pieces.

Domain: - social setting; counting numbers.

Stage II: The Halving Algorithm Stage

Operator: - partitioning in two; halving algorithm; no notion of equality.

Algorithm: - repeated dichotomies.

Domain: - one-half and other unit fractions whose denominate numbers are powers of two.

Stage III: The Evenness Stage

Operator: - equalness; congruency; halving algorithm becoming meaningful.

Algorithm: - halving algorithm; geometry transformations; extension of the halving algorithm to doubling any partition and adding two parts.

Domain: - unit fractions with even denominate numbers.

Stage IV: The Oddness Stage

Operator: - evenness; oddness; search for a new first move; use of the new first move; geometry transformations.

Algorithm: - exploratory measures; trial and error; counting; one-by-one procedure.

Domain: - all unit fractions.

Stage V: The Primeness Stage

Operator: - composition of numbers (Fundamental Theorem of Arithmetic); use of prime factors.

Algorithm: - multiplicative.

Domain: - all unit fractions.

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Table of Contents

Chapter Page

I. THE RESEARCH QUESTION 1

A. Conceptual Framework 1

The Rational Number Construct 1

B. Purpose of the Study 4

C. Definition of Terms 5

D. Design of the Study 5

E. Significance of the Study 6

F. Organization of the Dissertation 7

II. REVIEW OF THE RELATED LITERATURE 8

A. Introduction 8

B. The Partitioning Mechanism and Related Research 8

C. Rational Number Concepts Young Children Possess 13

D. Partitioning Behaviors 16

III. METHODOLOGY AND PROCEDURES 19

A. Research Methodology 19

B. The Pilot Study 24

C. Entry Into the School Setting 25

D. The Sample 26

E. The Tasks 28

F. The Clinical Setting 38

G. Clinical Session 40

H. Data Analysis 45

IV. PRESENTATION OF THE DATA 49

A. Selected Transcripts 50

Bro, (8;5) 50

Jou (8;5) 51

Hof (8;10) 52

Doz (9;8) 56

B. The Findings 57

The Partitioning Construct: Foundational Elements 57

Partitioning Behaviors: Categories and Description 58

Early Partitioning Behaviors 58

The Halving Mechanism 60

Rectangular Figure: Thirds 62

Rectangular Figure: Fifths 64

Circular Figure: Thirds 65

Circular Figure: Fifths 68

Circular Figure: Ninths 70

The Selected Figures 70

Structural Variances 71

Difficulty Levels 72

Stages of Partitioning Capabilities 78

Stage I. The Sharing Stage 78

Stage II: The Halving Algorithm Stage 80

Stage III. The Evenness Stage 83

Stage IV: The Oddness Stage 86

Stage V: The Primeness Stage 91

Conclusion 93

V. SUMMARY, DISCUSSION, AND RECOMMENDATIONS 95

A. Summary of the Investigation 95

The Sample 95

The Tasks 96

Conclusions 97

Proposition I 98

Proposition II 98

B. Discussion 101

The Research Methodology 101

The Findings 103

Development of Partitioning Mechanism: the Proposed Stages 104

Difficulty Level of the Selected Geometric Regions 105

Dominance of the Halving Mechanism ..106

Order of Achievement of Unit Fraction Partitioning Capability 106

The Significance of the Stages 107

C. Recommendations for Further Research 108

D. Concluding Statement 110

BIBLIOGRAPHY 111

APPENDIX A. LETTER TO PARENTS 119

APPENDIX B. DATA SHEET 121

APPENDIX C. SELECTED TRANSCRIPTS 125

List of Tables

Table Page

I Rectangular Regions: Successful and Unsuccessful Partitioning by Grade 73

II Circular Region: Successful and Unsuccessful Partitioning by Grade 75

III Rectangular and Circular Regions: Percentage of Successful Partitions Achieved, Halves and Fourths, Grade Levels 77

IV Rectangular and Circular Regions: Percentage of Successful and Unsuccessful Partitions Achieved, Thirds and Fifths, Grade Levels 79

List of Figures

Figure Page

1. The Rational Number Construct 3

2. Information Flow 22

3. The Clinical Setting 39

4. Early Partitioning Behaviors: Circular and Rectangular Regions 59

5. Partitioning Behaviors: Rio (5;0); Lon (5;6) 82

6. Partitioning Behaviors: Eighths and Sixteenths, Circular and Rectangular Regions 83

7. Partitioning Behaviors: Fifths and Sevenths, Circular and Rectangular Regions 90

8. Partitioning in Eighteenths: Circular Region 93

I. THE RESEARCH QUESTION

A. Conceptual Framework

A working knowledge of rational numbers is a realistic goal; however, recent national and provincial assessments (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980 a, 1980 b; Olson, Sawada, & Sigurdson, 1979; Robitaille & Sherill, 1979) reveal that far too many students fail to achieve this goal. The question arises as to why this is so. An examination of the rational number construct may shed light on possible answers to this query.

The Rational Number Construct

Among the reported attempts made to delineate the cognitive aspects of rational numbers (Hartung, 1958; Kieren, 1976, 1980 a, 1980 b; Rappaport, 1962; Streefland, 1978; Usiskin, 1979), the work of Thomas Kieren has been the most comprehensive. Kieren views the system of rational numbers as a complex construct with many representations arising out of real life situations.

Rational numbers are seen to have a kind of dual existence (Kieren, 1980 b). In one appearance they are quantities which can be added (e.g. One-fourth cookie and one-half cookie is as much as three-fourth cookie.), while in another they are functions (or relationships between quantities) which can be compared and thus are multiplicative (e.g. One-fourth of one-half of a candy bar is one-eighth of the candy bar.). Because of these two interpretations, rational numbers are means of describing different real life situations.

In unraveling the fabric of the rational number construct, one can identify various strands. Kieren (1980 a, 1980 b) presents four dominant strands or subconstructs, namely: measures, quotients, operators, and ratios (see Figure 1). A fifth subconstruct is the part-whole relationship which is the "fiber from which the rational number language is woven". In this function, the part-whole subconstruct is related to each of the other four subconstructs.

In rational number learning, the goal of any program ought to be to provide appropriate experiences in the various subconstructs--experiences which are in consonance with the maturational level of the person--so that the concept may eventually emerge as a coherent system in the mind of the young adult.

To develop a proper learning sequence, a knowledge of the mathematical and cognitive foundations of the rational number construct is necessary but not sufficient; required as well is the knowledge of how the concept develops in children and young adults.

In attempting to understand the genesis of the concept in young children, one must study the elemental mechanisms which are the precursors of the main subconstructs. Two constructive mechanisms which function as foundational elements of the rational number construct are partitioning

The Rational Number Construct

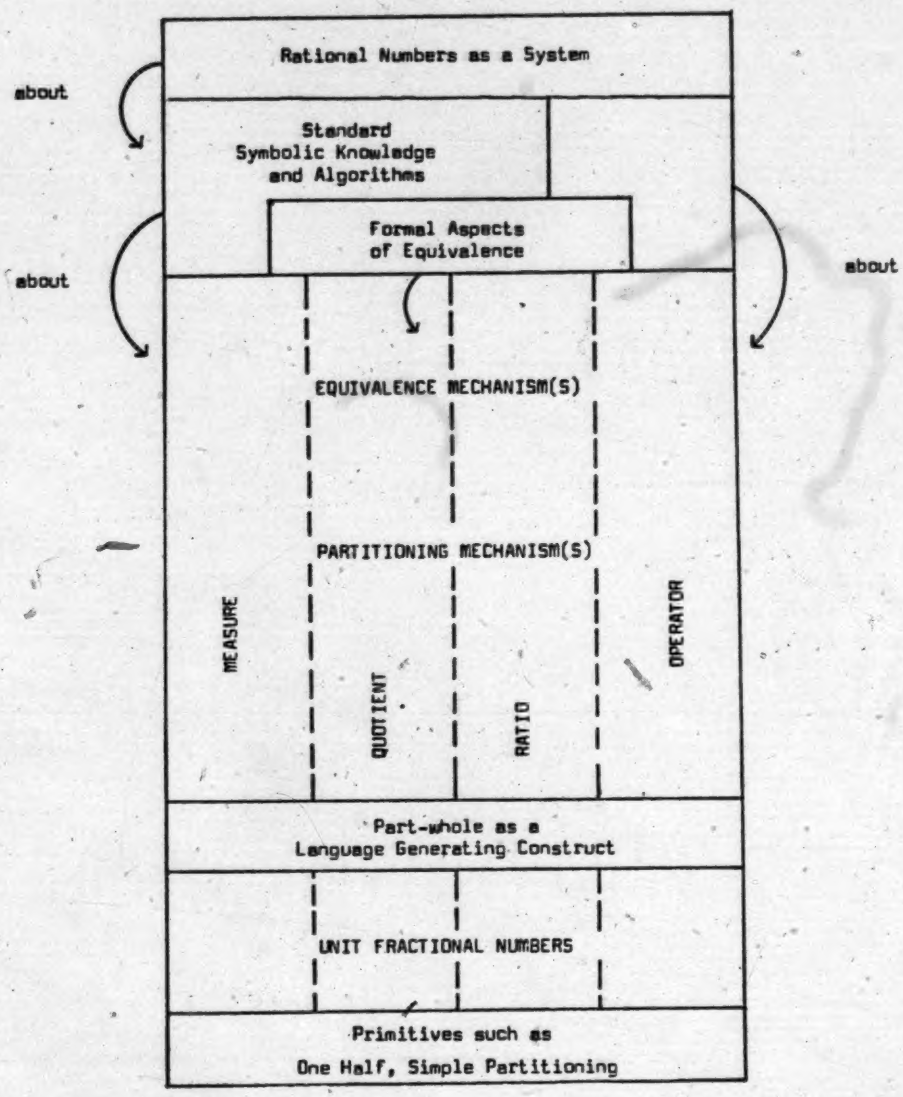

Rational Numbers as a System
about
Standard Symbolic Knowledge and Algorithms
Formal Aspects of Equivalence
about
about
EQUIVALENCE MECHANISM(S)
PARTITIONING MECHANISM(S)
MEASURE
QUOTIENT
RATIO
OPERATOR
Part-whole as a Language Generating Construct
UNIT FRACTIONAL NUMBERS
Primitives such as
One Half, Simple Partitioning

Figure 1

Kieren, T. E. (1980 c).

and equivalence (Kieren, 1980 c). What specifically is the role of these mechanisms in the development of the rational number construct?

For the present study, the focus is on the partitioning mechanism as it relates to rational number development. Questions asked are: What are the capabilities which undergird the partitioning construct? Are there observable behaviors exhibited by young children which could designate phases in the development of the mechanism? When is mastery of the partitioning mechanism evident?

B. Purpose of the Study

This study sought to understand the kinds of techniques that young children (aged 5 to 9 years) use when involved in solving selected partitioning problems.

The specific objective of the study was to describe and interpret the partitioning behaviors of young children as they manipulated given materials while solving selected partitioning problems.

Subsidiary questions related to the main objective were:

a. What categories of behaviors do children exhibit when solving partitioning problems?

b. Is there a hierarchy of behaviors within the partitioning realm?

c. What does it mean to say that a child has mastered the partitioning mechanism?

C. Definition of Terms

Rational Number

Any number that can be expressed as a ratio of two integers (whole numbers), providing the second number is not zero. In symbols, a number that can be put in the form a/b, where a and b are integers and b does not equal 0 (Marks, 1971, p. 124).

Partitioning

The subdivision of a discrete or continuous quantity into equal parts.

Partitioning Problem

Problems which involve partitioning a designated quantity or region into an indicated number of parts.

Partitioning Behavior

The method of procedure and verbal expressions of a child manifest while partitioning quantities or regions.

D. Design of the Study

The following is a brief exposition of the research design. A detailed explanation is presented in Chapter III.

The discovery-orientation of the study necessitated that a research technique within a discovery paradigm be employed. Accordingly, a clinical interaction technique was designed for the study. Modeled on Piaget's clinical interview, the research method involved the researcher in clinical sessions with individual children as a series of case studies.

Following the method of theoretical sampling (Glaser & Strauss, 1967), the selection of respondents continued throughout the period of data collection. Data analysis was distinguished by three stages, commencing at the initial time of data collection and enduring for several months after all the data were collected.

The sample of 43 children in Kindergarten, grades one, two, and three, was drawn from a primary school in an Albertan town. The respondents, all acquainted with the researcher, were selected on the basis that they would freely interact during a clinical session.

The clinical sessions were videotaped and audiotaped and later transcribed.

E. Significance of the Study

The study sought to gain insights into the construction of the rational number concept in children. It specifically focused on the development of the constructive partitioning mechanism and how it relates to the concept of rational number.

An understanding of rational number development can have far reaching effects on the scope and sequence of a primary mathematics program.

F. Organization of the Dissertation

A review of the relevant literature will be presented in Chapter II. It includes an examination of the partitioning mechanism and related research, partitioning behaviors, and rational number concepts young children possess.

Chapter III contains a description of the research design, the five partitioning tasks, the data collection, and analysis process.

The findings of the study are presented in Chapter IV.

The concluding chapter, Chapter V, includes a summary and discussion of the major findings and recommendations for further research.

II. REVIEW OF THE RELATED LITERATURE

A. Introduction

Historically, the study of rational numbers has occupied a secure place in the mathematics curriculum (Kieren, 1976). Despite this fact, to this date, there is no consensus concerning sequence (Firl, 1977; Payne, 1980; Peck & Jencks, 1977) and the teaching procedures (Hartung, 1958; Streefland, 1978) for rational number learning; moreover, the concept remains a difficult one for many learners (Ekanstan, 1977; Vergnaud, 1979). Focusing on partitioning may provide a fresh perspective on these long standing concerns.

B. The Partitioning Mechanism and Related Research

Partitioning is postulated to be one of the constructive mechanisms which leads to the development of the rational number concept. Kieren (1980 a) speculates that "partitioning may play the same role in the development of rational number constructs that counting does vis a vis the natural numbers" (p.22). One can ask, how is counting related to the development of natural number?

In analysing the young child's understanding of number, Gelman & Gallistel (1978) make a distinction between the process of reasoning about number in a general way (e.g., the number two as an abstract reality) and the process of abstracting a number from reality (e.g., the number two or numerosity two). Whereas the former process is as yet foreign to preschool children, the latter is within the realm of their ability to think about numbers.

Gelman & Gallistel (1978) state that

> The normal preschool child's numerical reasoning appears closely tied to the procedure that generates the mental entities that he manipulates when he reasons numerically. And that procedure is counting. Thus, the child's arithmetic system is strongly shaped by the mental entities with which it deals, namely, the representations of numerosity that may be obtained by counting (p.184).

This viewpoint on the development of number in children is in marked contrast to the viewpoint of Piaget (1952) who contends that the primitive basis for ascertaining equal sets is by one-to-one correspondence. For Piaget, the function of counting serves only as a kind of reinforcement for the concept of number as a quantitative entity.

Piaget states that

> At the point at which correspondence becomes quantifying, thereby giving rise to the beginnings of equivalence, counting aloud may, no doubt, hasten the process of evolution. Our only contention is that the process is not begun by numerals as such (p.64).

For Gelman & Gallistel, the one-to-one correspondence technique does not mark the inception of numerical equivalence but, rather, it is employed at a later stage in the use of reasoning principles.

This brief discussion on the development of number in young children leads one to question how and when the concept of rational number is first conceived by children.

Considering the proposition that children's "natural" act of counting is the mechanism which they employ to first grasp the concept of numerosity (Gelman & Gallistel, 1978), one is led to look at the behaviors of children for a precursive action to the development of rational numbers. Kieren (1980 a, 1980 c) has identified partitioning as one such plausible primitive experience.

While there have been recent studies concerning the various interpretations of rational numbers [Measurement: (Piaget, Inhelder, & Szeminska, 1960; Babcock, 1976; Sambo, 1980), *Ratio and Proportion*: (Hart, 1978; Karplus & Peterson, 1970; Karplus et al., 1974; Muller, 1979; Noelting, 1978; Rouchier, 1980; Van den Brink & Streefland, 1979), *Operator*: (Ganson & Kieren, 1980; Kieren & Nelson, 1978; Kieren & Southwell, 1978, 1979, 1980)], there has been scant attention paid to the partitioning mechanism as related to the development of rational number other than the studies of Piaget, Szeminska, & Inhelder (1960).

Piaget and his colleagues observed the partitioning behaviors of young children while studying the development of the fraction concept within the context of subdivision of areas. Thus, the materials used were of a continuous nature and consisted of clay and paper 'cakes' of different sizes and shapes. Prior to the successful equidivision of a whole into a specified number of parts (halves, thirds, fourths, fifths, and sixths) a sequence of behaviors were noted: general fragmentation; approximate equal sharing without exhaustive division (i.e., some cake 'left'); confusion on the number of cuts with the required number of parts.

Piaget et al. report that the ability to partition a whole into fourths follows mastery of partitioning into halves. Fourths are generally arrived at by carrying out two successive dichotomies (i.e., dividing the whole into two equal parts and then each part in half). Trichotomy is resolved next, an action which is thought to be considerably more difficult than dichotomy. Finally, the ability is transferred to subdivision of a whole into fifths and sixths.

Hiebert & Tonnessen (1978) attempted to replicate and extend the work of Piaget et al. (1960). Besides the physical representations of continuous quantities (length and area) a discrete quantity representation (set/subset) was included. Their findings concur with Piaget et al. on the continuous quantity tasks of length and area. The set/subset discrete task was found to be considerably easier than the continuous cases. A one-by-one strategy was frequently used to solve the set/subset partitioning problem.

In other instances, partitioning behaviors have been examined in the context of the division process. Zweng (1964) conducted a study with second-grade children which investigated the comparative difficulty of solving four types of division problems. One aspect of the study was an investigation of the methods the subjects used to solve partitive division problems. Zweng identified two strategies which she labelled sharing and grouping. If the same number of elements was assigned to each of the required subsets but all elements were not used on the first assignment, the method of solution was called sharing. If all elements were assigned to the groups on the first try, whether the groups were the same or not (e.g. 18 elements: groups of 6,6,6 or 7,5,6), the method was called grouping. Zweng reports that over twice as many problems were solved by grouping as by sharing. One-by-one correspondence was seldom used.

A study conducted by Weiland (1977) addressed itself to the partitioning behaviors of young children during discovery-oriented instructional settings in division using base ten blocks. The sample consisted of 21 seven-year-old children who understood numeration and were familiar with the set of materials. Of interest is the finding that twelve of the 21 subjects used the strategy or argument of halving to solve problems whose divisor was a power of two. [Twenty-seven of the 66 problems given to each child were of this type.] Wieland describes the children's strategy thus:

> Children using the argument took half of the dividend to divide by two, one-quarter or half of a half to divide by 4, and one half of a half of a half to divide by 8 (p. 110).

Nelson & Sawada (1975) studied the problem solving behaviors of young children aged 3 to 8. One of the paired problem settings presented to each child was a partition and measurement problem involving discrete objects. Nelson & Sawada report that "only a few children at the upper age levels showed that they had mastered the partitioning process" (p.37).

C. Rational Number Concepts Young Children Possess

Historically, the system of rational numbers was developed because of the need to describe parts of a unit or of a group. For example, questions like the following could not be answered with a member of the set of whole numbers: What part of the chocolate bar did you eat? If 2 pizzas are shared equally among 5 friends, how much does each person get? In answering the last question, one can see that the rational number 2/5 results from the division of one whole number by another, that is, 2 divided by 5.

Do children grasp the concept of rational numbers as quantitative entities at the time they comprehend division of whole numbers? Research shows that children know some rational numbers at an earlier time (Campbell, 1975; Gunderson, 1940; Gunderson & Gunderson, 1957, 1959; Polkinghorne, 1935; Suydam & Weaver, 1975; Williams, 1965).

Polkinghorne (1935) tested 266 children from kindergarten and grades one, two, and three to ascertain what fraction concepts they had and when and how the children acquired them. The results obtained demonstrated that the children had certain concepts of fractions: unit fractions were understood better than proper fractions which are not unit fractions and improper fractions; very few children knew anything about identification of fractions; and only one child showed some knowledge about equivalent fractions. The children cited experiences at home and at school to indicate how they acquired some fraction concepts. Polkinghorne reports an instance of a child defining half and demonstrating half a set (4 pencils) and concludes that the child has "perfect understanding" of half.

Gunderson & Gunderson (1957) interviewed grade two children (7 year olds) and questioned them about fractional parts. Flannel circles (pies), pre-cut into fractional parts, were the materials used. Two lessons on fractions preceded the interviews. A first lesson involved the children folding paper circles in half, quarters, and eighths while during a second lesson they used paper circles pre-marked in thirds and sixths. It is reported that in the course of the lessons, the fractional terms half, fourth, quarter, third, and sixth were elicited from the children. The 15-minute interviews focused on comparing the fractional parts taught in the two lessons. Questions comparing unit fractions, multiple fractions, and whole and mixed numbers were asked. The children were requested to defend their answers and to think aloud as they manipulated the flannel pieces. Gunderson & Gunderson report that "the children had no difficulty in identifying fractions or fractional parts" (p. 169). They concluded that their subjects "showed a good understanding of fractions using manipulative materials" and that they could "...obviously profit from planned systematic instruction in the meaning and use of fractions" (p.173).

A study of number concepts held by 7 year olds was conducted by Gunderson (1940) and replicated twenty years later (Gunderson & Gunderson, 1959). From the more recent study, Gunderson & Gunderson report that the children showed "an understanding or insight into fractions far greater than that revealed by the pupils in the earlier study" (p.185). This greater knowledge of fractions is attributed to

> an early teaching of fraction meaning or concepts wherein children through the use of fractional parts or cut-outs discover fraction concepts by superimposing one fractional part over another to see how they compare in size (p. 185).

Williams (1965) conducted a study of 595 kindergarten children during the first month of schooling to assess their mathematical concepts, skills, and abilities. Williams reports that 25.8% of the sample knew "half of a whole" and 28.7% understood "half of a group of four dots".

A study by Campbell (1975) addressed itself to rational number knowledge, specifically the fractions one-half, one-third, and one-fourth under three interpretations: part of a whole; part of a set; and operation of division. Concrete and semi-concrete tasks were used. A sample of five, six, and seven-year-olds was tested prior to receiving formal instruction on the topic. The results were: half a whole was better understood than any other interpretation while one-third was understood better than one-fourth. The size of elements seemed to affect a child's understanding of one-third or one-fourth of a set but did not appear to affect understanding of one-half of a set. Most children showed understanding of how many halves, thirds, and fourths are in a whole when concrete tasks were employed.

Suydam & Weaver (1975) surveyed the literature to discover what had been ascertained about the mathematical knowledge children have upon entering school. Regarding fractional knowledge, they make a single imprecise statement: "Most [children] have some knowledge about coins, times, and other measures; about simple fractional concepts; and about geometric shapes" (p. 49).

The studies by Williams (1965) and Campbell (1975) provide evidence that young children possess some knowledge of fractions prior to schooling, whereas, the studies by Gunderson (1959), Gunderson & Gunderson (1957), and Polkinghorne (1935) demonstrate that primary children can learn some fraction concepts particularly if taught using concrete representations.

D. Partitioning Behaviors

Kieren (1980 c) interprets the act of partitioning under four aspects.

1. A classification or allocation based on the criterion of 'evenness' or 'just enough'.
2. The social act of sharing. Systematic partitioning techniques such as 'dealing out' can be generated from sharing activities.

3. The language describing the act and results of partitioning.
4. The attachment of the parts to measure or number.

In addition, Kieren describes five sets of expected partitioning behaviors in children. These are:

a. Separation: - dividing into the appropriate number of sets with classification on bases other than number.

b. Evenness: - some subsets even, some not; an evenness criteria; dividing by two by matching.

c. Algorithmic Partitioning: - 'dealing' behaviors by ones or 'mixed' dealing behaviors.

d. Partitioning and Number: - relating 'evenness' and number or size of the finished partitions; realizing that quantity is preserved under partitioning; generalized partitioning.

e. Advanced Partitioning: - repeated partitioning; related number and size of partitions; given a partition, change it by adding to the number of partitions or reducing the number; given a partition, see another within it (Kieren, 1980 c, pp. 5-6).

The partitioning act, particularly when involving continuous quantities, can be a challenge for the young child (Piaget et al., 1960). Assigning a number value to a partition (e.g., part of a cake) marks a departure from whole number quantification and extends into the system of rational numbers. At what age are children able to intuitively make this mental 'jump'? What physical representation enables them to first succeed in this transfer? In other words, when and how does the part-whole and part-part relationships emerge at an informal level?

To find answers to these questions, the study involved children partitioning materials varying in substantive nature.

III. METHODOLOGY AND PROCEDURES

A. Research Methodology

A clinical interaction technique used within a discovery paradigm was deemed the most appropriate research methodology for the objectives of the study.

The discovery paradigm upon which the investigation is based is that presented by Sawada (1980). Sawada explains the purpose of the paradigm in the following words:

> The paradigm is a paradigm for discovery. It is a generalized scheme for designing a sequence of sensitive interactive encounters between researcher and child for the purpose of generating concepts that can play foundational roles in understanding the child's acquisition of mathematical ideas (p.2).

The "sensitive interactive encounters" designed for the present study are modeled on Piaget's clinical interview and are for the purpose of discovery of insights rather than for the verification of hypotheses. Thus, the validity as opposed to the generalizability of the insights is highlighted.

Piaget's "methode clinique" consists of three research techniques, namely; pure observation, questioning, and clinical examination.

According to Piaget, "Observation must be at once the starting point of all research dealing with child thought and also the final control in the experiments it has inspired" (p.4). Focusing on the child's thought, observation is directed to "the spontaneous questions of children". The questions posed to the child are "determined on matter and in form, by the spontaneous questions asked by the children...". A clinical examination technique allows the researcher to probe in order to "judge" the child's expressions (Piaget, 1929, pp. 4-8).

In essence, the clinical method is "dependent on direct observation, in the sense that the good practitioner lets himself be led, though always in control, and takes account of the whole mental context..." (Piaget, 1929, p.8).

Ginsburg (1981) states that the clinical interview designed by Piaget is an appropriate methodology for three goals of research in mathematical thinking, namely: the *discovery* and *identification* of cognitive activities and mathematical *competence*.

Addressing himself to the discovery of cognitive processes, Ginsburg (1981) describes the clinical interview in the following manner:

> ...The clinical interview procedure begins with (a) *a task*, which is (b) *open ended*. The examiner then asks further questions in (c) a *contingent* manner, and requests a good deal of *reflection* on the part of the subject (p.6).

Ginsburg reiterates Piaget's descriptive components of the clinical interview and points out their relevance for identifying and describing mathematical cognitive structures. In Ginsburg's words, the clinical sub-goals are:

> ...to facilitate rich verbalization which may shed light on underlying process; ...to check verbal reports and clarify ambiguous statements; and, ...the method uses procedures aimed at testing alternative hypotheses concerning underlying

processes (p.7).

In accord with the goals of the discovery paradigm (Sawada, 1980) and patterned on Piaget's clinical interview (Piaget, 1929), a clinical interaction technique was employed in the present investigation.

Within the study, the term clinical denotes two essential characteristics of the methodology: direct observation on a one-to-one basis. By direct observation is meant a focusing of the researcher's attention on the respondent (in this case, the child) for a specific purpose, noting overt actions and verbal expressions, and recording these or significant aspects in a suitable manner. Such observations give birth to percepts which serve to guide the researcher in directing the flow of information during the clinical interaction.

The term interaction originates from two sources: a) questioning which results in verbal interchanges, and (b) the tasks which require manipulation of materials as a method of solution. Information grows out of the interaction which is characterized by a ten-directional flow: the researcher, child, and task inter-react; the researcher inter-reacts with the interaction between child and task; the child inter-reacts with the interaction between researcher and task (see Figure 2).

By virtue of the interactional process, both researcher and child contribute to the data in a critical way. The child's behavioral response to the task is the primary data

Information Flow

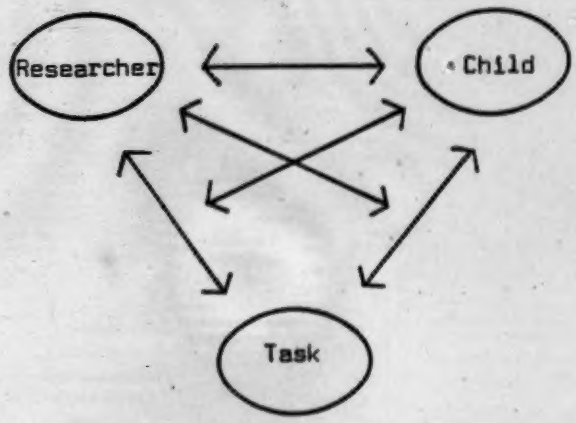


Figure 2

base. Reacting to the child's response, the researcher formulates subsequent questions and directives and thus shapes and directs the flow of information. The clinical process moves in cyclical fashion as researcher and child interact with the task situation and respond to each other. Additional cyclical movements follow a spiral path and evoke more challenges and probing to a greater depth.

The flow of information is also affected by the personological characteristics of the researcher. Descriptions of the qualitative researcher found in the literature reveal dominant characteristics. Among others are cited the sensitivity of the researcher (Good, 1959; Wilson, 1977), the ability to listen (Piaget, 1929; Posner & Gertzog, 1979), being skilled in the art of questioning

(Ginsburg, 1981; Piaget, 1929), and a capacity for empathic understanding (Posner & Gertzog, 1977). Moreover, in order to "tune-in" to the thought process of the child, the researcher must be able to receive and retain pertinent information; to reflect and react upon it in order to shape and direct the flow of information throughout the interaction; to accept and challenge the child while searching for clues which will serve as insightful leads in coming to understand the child's thought and behavior relative to the research question.

The clinical interaction is characterized by a flexibility in the questioning process which allows for individual differences in understanding the task to which children are to react (Brownell, 1944; Piaget, 1929).

In the present study, only the initial questions related to each partition problem were standardized. Subsequent questions were probing to elicit reasons for certain behaviors.

The questioning followed a general pattern but varied, as did the numerical difficulty of the problem situation, depending on the behavior of the child.

The methodology allowed for in-depth probing; it allowed for follow through with any peculiar behavior of a child. It permitted me, as researcher, to expand on the questioning or alter the problem situation until a point of closure was reached. Moreover, the procedural nature of the tasks admitted a continuously modifiable technique which was another way for maximizing validity. The procedural nature of the clinical interaction and the substantive nature of the tasks were considered to be methods which provided valid information on each child.

B. The Pilot Study

Training in clinical work with young children preceded the pilot study which was conducted over a period of seven months.

During February and March, 1980, a total of 23 children from a private and a public school kindergarten were observed while they solved a partition and a measurement division problem. These were the Animal Groups Problem and the Cargo Groups Problem used by Bourgeois (1976). In June, 1980, a subset of 14 children from the original sample were observed and interviewed while solving four of the partitioning problems described in this study. All the interaction sessions were audiotaped and subsequently transcribed.

A final pilot session, in September, 1980, involved eight children from grades one, two, three, and one child from grade four. Five of these interaction sessions were videotaped.

The purpose of these pilot sessions was four-fold: to gain further experiences in conducting clinical interactions with young children; to field test and refine the tasks, questioning techniques, and procedures; to borrow from the language of children in formulating interview questions; and, to refine recording procedures.

C. Entry Into the School Setting

Due to the type of research methodology, it was judged important to spend some time in the school setting prior to engaging in clinical interactions so that the children would come to know me as a teacher. Thus, a week was spent in the school observing and interacting with individuals and small groups of children in the four classrooms from which the sample was eventually drawn.

The location of the school is in a northwest Albertan town with a population of 6000. The occupation of the majority of the working population is connected in one way or another with mining and oil industries.

The school is one of two elementary schools in the town with an enrollment of 248 pupils in 6 kindergarten (half day classes), 3 grade one, 4 grade two, and 3 grade three classes.

Four participating teachers, kindergarten to grade three, were selected by the principal subsequent to an explanation of the purpose and procedures of the study. The teachers agreed not to teach fractions during the time of data collection and avoided querying the children about the clinical experience. This latter precaution prevented involuntary or undue discussion among the children.

The specific content and nature of the clinical session were not revealed to the teachers at the outset.However, they were told that they would be welcome to familiarize themselves with the materials and view some tapes during the final days of my stay among them. This was accomplished by a formal presentation to the teachers during a regular staff meeting held on my second last day in the school. A pleasant and memorable experience made possible by the warm reception and full cooperation of the entire school community throughout the seven weeks was brought to a climactic close by the teachers' enthusiastic reaction to the study question, materials, and procedures together with an expressed eagerness to know the findings.

D. The Sample

The forty-three children who eventually formed the sample were from kindergarten and the first three grades in a primary school of the Yellowhead School Division, Alberta.

In accordance with the discovery paradigm, the selection of respondents deviated markedly from traditional sampling procedures. Thus, no a priori decision was made in regard to sample size; rather, the sampling process within age, grade, and sex variables was ongoing throughout the period of data collection. This allowed latitude for extensive study of any behavior of any child at any age level.

During the "week of entry" into the school, a pool of children was identified for possible inclusion in the study. Factors deemed important in the selection of the initial pool and subsequent additions were: English as a first language; tendency to verbalize freely; no apparent learning disability; no evident health problem. The aim was to interact with a child at a time when he or she was likely to perform optimally.

Throughout the time of data collection, respondents were chosen each day. The sampling continued until sufficient key insights had emerged and a theoretically satisfying degree of conceptual integrity connected the various insights.

At the close of the clinical sessions, the sample of 22 girls and 21 boys included 8 kindergarten, 8 grade one, 12 grade two, and 15 grade three children. Ages of the children ranged from 4 years, 11 months to 9 years, 8 months with a mean age of 7.12 years.

Scores attained by the children on Standardized Readiness and Achievement Tests were obtained from school records following completion of the clinical sessions. Also, at this time, teachers were asked to rate the children on their current performance in mathematics. These ratings designated 12 children as high average, 22 as average or better, 7 as below average, and 2 as very low on mathematics achievement.

In summary, the sampling procedure can be described as three-dimensional:

a. children across age, grade, and sex (described above),
b. problems within persons, and
c. behaviors within problems. (For an explanation of (b) and (c) see description of methodology, p. 21.)

Parental consent was obtained for the children to take part in the study. A sample letter sent to the parents is included in Appendix A.

E. The Tasks

Five partitioning problems were devised for the study. The characteristics of "good" mathematics problems for young children elaborated by Nelson & Sawada (1975) were considered when producing the different tasks. Furthermore, the following five variables were taken into account:

1. Substantial Nature:
    a. discrete (Carton-Truck Problem)
    b. discrete set with elements divisible (Cookie Problem)
    c. continuous (Cake Problem)
    d. discrete set with subsets separable (Boxed-Candy Problem)
    e. continuous but divisible (Chocolate Bar Problem).

2. Procedural Nature:

Each task admitted a continuously modifiable partitioning technique.

3. Amount of 'Noise':

An attempt was made to eliminate qualitative distractions. (e.g., similar cartons, cookies, plates, and candies were chosen.)

4. Number of Partitions:

Two and powers of two such as four, eight, and sixteen; three and its multiples such as six, nine, and twelve; five and its multiples such as ten and fifteen; seven.

5. Number of Elements:

Capable of being evenly partitioned; not evenly partitioned.

The problem settings were seen to be within the experience of the child, whether in real life situations or while at play.

Manipulation of materials, with or without verbal expression, as a method of solution was deemed to be the mode which maximized the manifestation of the child's mathematical knowledge related to the task. That the context of a concrete method of solution is conducive to the young child's exhibiting his or her mathematical capability is suggested by Nelson (1979) as his research with young children bears out.

The five problems are described below.

Problem 1. Carton-Truck Problem

The Carton-Truck Problem
Task situation: 3 trucks, 15 cartons

The Carton-Truck Problem
Task situation: 3 trucks, 17 cartons

The Carton-Truck Problem was modeled on that employed by Bana (1977). The materials consisted of three toy trucks and a set of 'cartons'. The child was asked to load the cartons onto the trucks so that each truck had the same load to carry.

The trucks, 1 green, 1 blue, and 1 red, were 16 cm in length with a loading space capacity of 10 cm in length, 5.7 cm in width, and 1.7 cm in depth. The cartons were grey wooden blocks with dimensions 2.5 cm by 1.7 cm. Depending on how they were positioned in a truck, 6, 8, 10, or 12 cartons could be loaded per layer onto each truck.

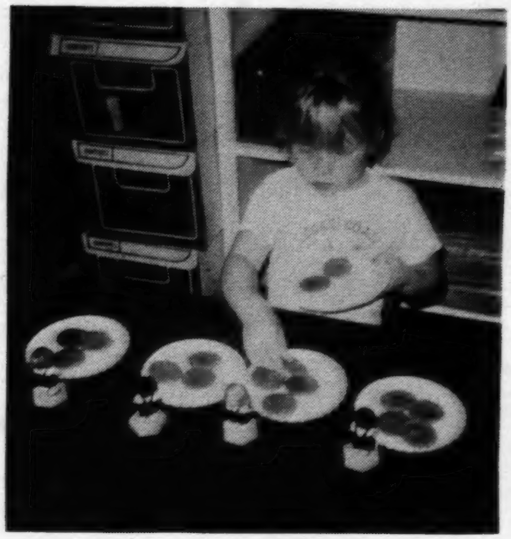
The Carton-Truck Problem was presented to each child under two different mathematical representations. One involved a set of cartons whose cardinal property was evenly divisible by the number of trucks (e.g., 3 trucks, 15 cartons). Another mode made use of a set of cartons whose cardinal property was not evenly divisible by the number of trucks (e.g., 3 trucks, 17 cartons). When the number of cartons was changed, the child could see how many cartons were added to the set.

Problem 2. Cookie Problem

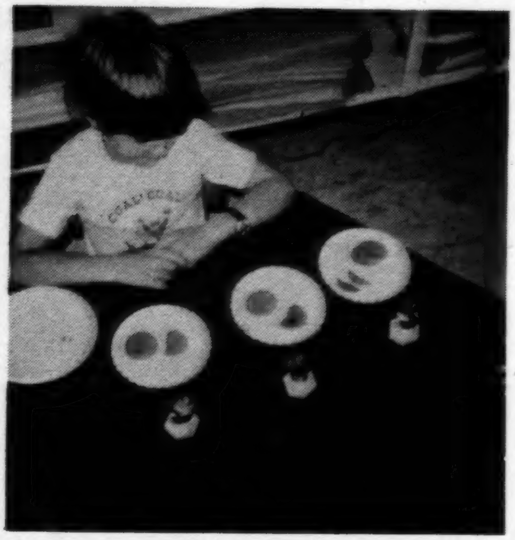
The materials for the Cookie Problem consisted of a set of dolls, paper plates, and cookies. The setting was a birthday party and the child was asked to give the cookies to the children at the party so that each person had the same amount.

Several sets of cookies were presented to each child; one was evenly divisible by the number of dolls in view (e.g., 4 dolls, 16 cookies) and the others divided unevenly (e.g., 4 dolls, 10 cookies; 4 dolls, 9 cookies; 3 dolls, 10 cookies; 3 dolls, 8 cookies).

Similar to the Carton-Truck Problem, the child could see that cookies were either added to or removed from the plate as the mathematical aspect of the problem was modified.



The Cookie Problem
Task situation: 4 children, 16 cookies



The Cookie Problem
Task situation: 3 children, 8 cookies

Each miniature doll stood on a wooden base to a height of 9 cm. The paper plates were 15 cm in diameter while a larger serving plate was 17 cm in diameter. The cookies were a commercial brand of gingersnap, each 5 cm in diameter.

The materials for this problem embody the phenomenon of discrete sets with elements divisible.

Problem 3. Cake Problem

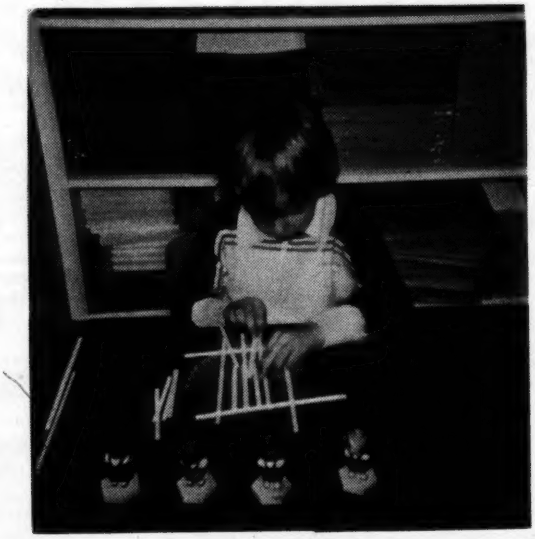
Rectangular and circular 'cakes' of various dimensions and 'sticks' to demonstrate cuts on the cake were the materials for the Cake Problem. The child was asked to demonstrate how he or she would cut the cake so that each person had as much.

The rectangular cakes were white styrofoam models, 5 cm in depth, and topped with a piece of plush gold rug glued to the surface. The dimensions of the several rectangular cakes were: 10 cm by 25 cm, 15 cm by 20 cm, 20 cm by 25 cm, and 2 cakes 10 cm by 15 cm. A circular cake, 20 cm in diameter and 2.4 cm in depth was constructed of similar materials as the rectangular cakes. A circular piece of rug, 16 cm in diameter, was also used within the context of this problem and was referred to as a 'giant cookie'. The 'sticks' which served to demonstrate cuts on the cakes and giant cookie were white plastic molding, 1 mm in thickness, cut in strips 0.5 cm wide and of lengths 10 cm, 19 cm, and 27 cm. The complete set of sticks was available for the child's use when partitioning each cake for any number of people.

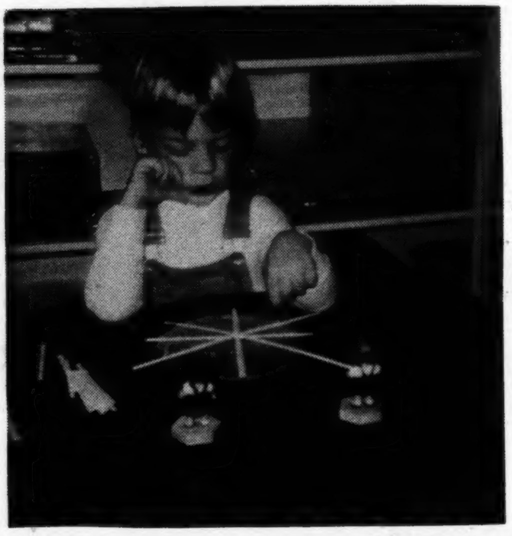
Generally, the child was first requested to partition the cake for 2 people, then for 4, 3, and 5 people. However, the order of presentation varied for some children and not everyone was asked to partition the cakes for the same groups of people. Older children were presented with more partitioning problems than younger ones. Other sequences for the number of required partitions were: 2, 3, 4, 5, 6 people; 4, 8, 16 people; 3, 6, 12 people; 5, 10, 20 people.

Rectangular and circular shaped cakes were selected for two main reasons:

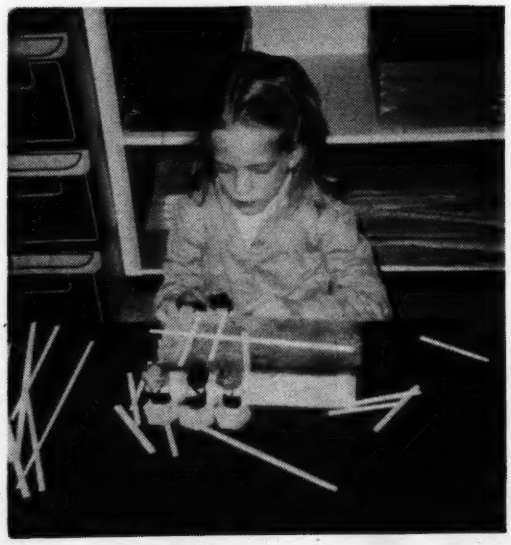
a. These were thought to be the shapes which children



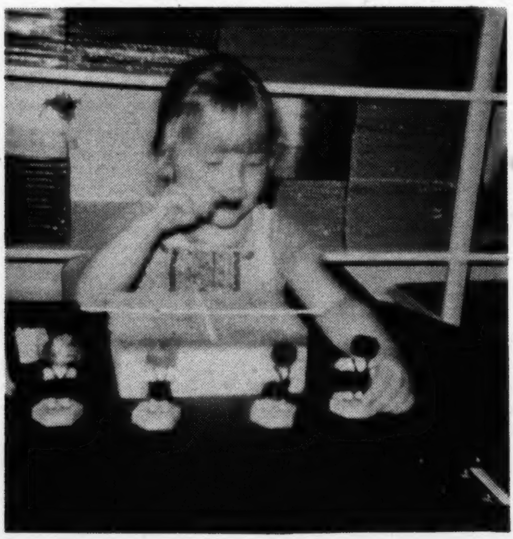
The Cake Problem
Task situation: 4 children, circular region



The Cake Problem
Task situation: 2 children, circular region



The Cake Problem
Task situation: 3 children, rectangular region



The Cake Problem
Task situation: 4 children, rectangular region

were more likely to have had experience in partitioning, or to have seen partitioned;

b. To successfully partition a rectangular and circular region in 2, 3, 4, 5, and more congruent pieces necessitates that the child have knowledge of distinguishing properties of the two geometric figures. To find out whether the two figures were equally challenging, and if not, which one was more difficult was a subordinate problem objective.

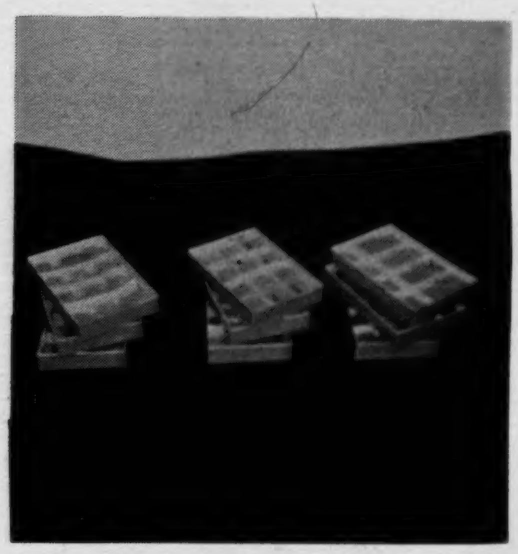
The circular region was partitioned first; that is, immediately following the Cookie Problem, the child was presented with the giant cookie and subsequently with the rectangular cakes. The round cake was generally employed for further partitioning of a circular region following work with the rectangular cakes or during a second interaction session.

The Cake Problem exemplified a partitive division problem involving a substance of a continuous nature.

Problem 4. Boxed-Candy Problem

Identical boxes sectioned into 12 compartments (3x4) were filled with wrapped caramel candies, one per compartment.

The child was asked to share the boxes of candy among a group of children (party setting) so that each person had as many boxes. The groups varied; for example, 4 dolls, 9 boxes; 4 dolls, 6 boxes; 3 dolls, 7 boxes.



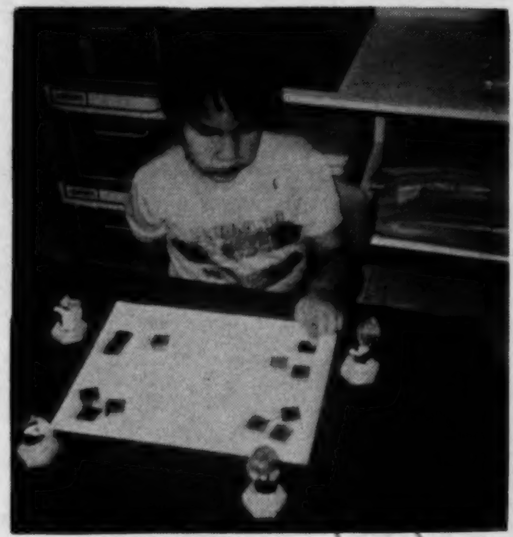
The Boxed-Candy Problem
Materials

Each box, 14.5 cm by 9.4 cm, was filled with candies which were kept in place by a transparent acetate cover.

The Boxed-Candy Problem, in essence, represented a set whose elements are discrete and separable (not continuous).

Problem 5. Chocolate Bar Problem

Square ceramic tiles, 2.5 cm, dark brown in color, were the materials for the Chocolate Bar Problem. The adhesive pile surface of Velcro nylon material was cemented to the bottom surface of each tile. The tiles could be fastened to a complementary piece (surface of tiny hooks) of Velcro tape



The Chocolate Bar Problem
Task situation: 4 children, 12 piece chocolate bar

The Chocolate Bar Problem
Task situation: 3 children, 12 piece chocolate bar

to make different sized chocolate bars, as for example, 2 strips of 4 tiles (a 2x4 bar) or 2 strips of 6 tiles (a 2x6 bar).

The child was asked to share the chocolate bar among the children (at the party) so that each person had an equal share.

The Chocolate Bar Problem was a model of a substance continuous in nature but divisible.

F. The Clinical Setting

The clinical sessions were conducted in a small but adequately sized room which ordinarily served the dual purpose of a storage area and the school nurse's office. Packaged paper and other classroom supplies filled shelves along two opposite walls while three large windows occupied the greater part of the connecting wall.

Two folding cots were aligned end to end along the narrow arm of the L-shaped room. On three occasions during clinical sessions, a sick child quietly entered the room and rested on a cot. An improvised movable partition served to 'isolate' the sick child from the respondent and the interaction went on uninterrupted. After a duration of twenty minutes or less, a teacher or parent came for the sick child.

A low trapezoid-shaped table and two small chairs occupied the center of the wider section of the room. This was the locus of the interaction and consequently, the materials and mechanical apparatus were strategically situated in close proximity to this central area. A floor plan of the room with arrangement of furniture and equipment are depicted in Figure 3.

The clinical sessions were registered on 1/2 inch VHS tapes, of two hour length, using a Panasonic Video Cassette Recorder (NV-8200) and a Panasonic Video Camera (WV-361P). The video recorder and TV monitor (9 inch screen) were positioned on shelves behind the respondent. The camera was

The Clinical Setting

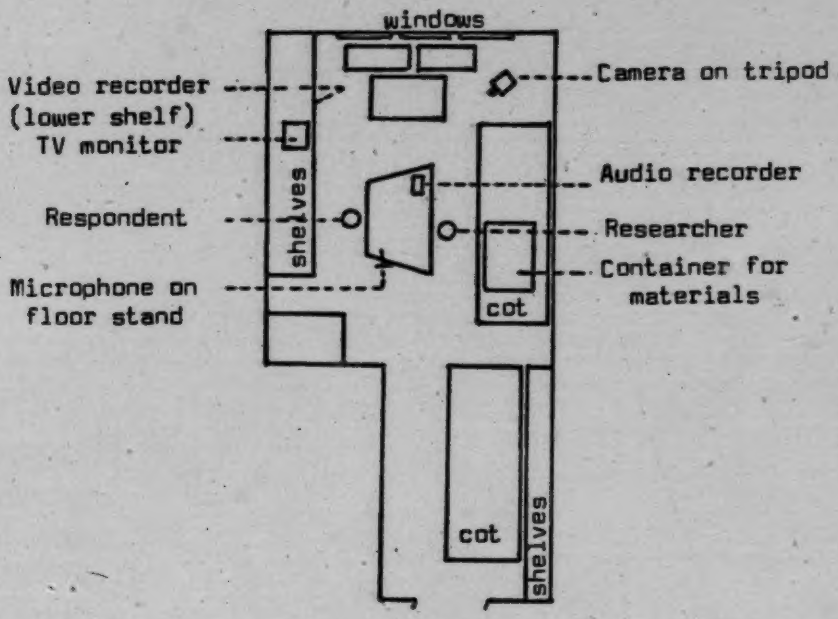

windows
Video recorder (lower shelf)
TV monitor
Respondent
Microphone on floor stand
shelves
Camera on tripod
Audio recorder
Researcher
Container for materials
cot
cot
shelves

Figure 3

focused on the child seated at the table while a Sony (F540) microphone rested on a floor stand close to the oblique side of the table on the child's right.

The sessions were also recorded on a Superscope Audio Recorder which was situated on the table at my right. Pre-numbered 60 minute tapes were on the table in readiness for a quick change. Operating the recording equipment soon became almost an automatic procedure and did not distract me from the activity and questioning whenever they were in progress. The majority of the children appeared to forget that they were being videotaped as they became interested and diligently occupied with the partitioning tasks.

The children, each in turn, were accompanied from their classroom to the assigned room and during this short walk, we engaged in light conversation which served to secure an informal atmosphere for the interaction session. Upon entering the room, the child was guided to the proper chair and the purpose of the mechanical apparatus was given. The function of the separate pieces of equipment--the camera, video recorder, TV monitor, audio recorder, and microphone --was briefly explained to the child with the promise that after we were finished using the materials, there would be an opportunity to view him/herself on TV. During this five minute viewing, the children manifested delight and satisfaction at thus seeing and hearing themselves.

G. Clinical Session

Following the brief explanation of the function of the mechanical apparatus, the clinical session began with the child comfortably seated.

The materials for the different problems were kept in a large suitcase situated to my left; readily available yet hidden from view of the child.

While the materials for the problem were placed on the table before the child, the problem setting was introduced and the purpose of each object was explained. During this time, the child was encouraged to manipulate the materials by such comments as: "Line up the trucks this way." "Would you like to give a plate to each child?"

In formulating questions throughout the clinical sessions, extreme care was taken to avoid difficult concepts in terminology. Samples of children's language obtained from the pilot studies were included. (e.g., "Is it a fair share?" "Are they all happy?" "Which one is sad?") No mathematical term was used unless the child had previously voiced it. For example, if the child had verbalized the terms third and quarter, questions like the following may have been subsequently posed: "A while ago, you called a piece, a third. Could this (e.g., a quarter of the cake) be a third?" "Why?" or "Why not?"

Initial questions were open-ended while subsequent questioning included probing to elicit reasons for certain behavior. These were:

a. Clarifying questions to obtain quantitative expressions of partitioning behaviors. (e.g., "How could this girl tell her mother how much cake she ate at the party?" "What could this boy say?" "Could he say that another way?")

b. Confirming questions to ascertain if the child understood the task. (e.g., "Is that a fair share?" "Why?" "Why not?" "Are they all happy?" "Which one is sad?")

c. Defining questions to get to the meaning of the verbal expressions. (e.g., "What does 'cutting in half' mean?" "What does that mean, 'a number that's even'?")

d. Introspective/retrospective questions to gain insights into the thought process of the child. (e.g., "Tell me what you're thinking as you're making those pieces." "Why is it that for three pieces you didn't use this stick?")

After a solution, the following command was frequently repeated: "Show me another way" [to solve the problem].

All behaviors and verbalizations were generally accepted without feedback as to appropriateness of response and the children usually manifested a sense of satisfaction with their achievement. For some children, it was apparent that some tasks were a first encounter. The expressed delight at discovering how to solve a problem after a series of attempts was indeed a pleasure to witness.

The probing was generally begun after the performance of the child had been assessed; that is, when it was decided that the child had demonstrated his or her capabilities regarding the tasks. Often, while in-depth questioning ensued, materials were returned to the table or a different sized cake would be used or the circular cake in lieu of the giant cookie. Changing the materials also helped to maintain a high degree of interest.

The procedures and initial questions for the five partitioning problems follow.

Carton-Truck Problem

1. THE DRIVERS OF THESE TRUCKS HAVE TO DELIVER SOME CARTONS TO DIFFERENT STORES. THE CARTONS HAVE TO BE LOADED ONTO THE TRUCKS. (Demonstrate by putting a carton onto a truck.) CAN YOU DO THIS SO THAT EACH TRUCK HAS AS MUCH?
2. When the child has completed the partitioning, say: SHOW ME ANOTHER WAY TO LOAD THE CARTONS SO THAT EACH TRUCK HAS AS MUCH.

Cookie Problem

1. Birthday party setting. THESE CHILDREN ARE AT A BIRTHDAY PARTY. (Place the dolls on the table.) ONE OF THE THINGS THEY HAVE TO EAT ARE COOKIES. THE COOKIES CAN BE PUT ON THESE PLATES. (Place a plate before each doll or have the child do so.) Offer a plate of cookies to the child. Ask: CAN YOU GIVE THE COOKIES TO THE CHILDREN SO THAT EACH PERSON HAS AS MUCH?
2. When the child has completed the partitioning, ask: DO THEY EACH HAVE AS MUCH?
3. Say: SHOW ME ANOTHER WAY TO GIVE THE COOKIES TO THE CHILDREN SO THAT THEY EACH HAVE AS MUCH.

Cake Problem

1. Say: A FAMILY HAS THIS CAKE TO SHARE. IF THERE ARE ONLY THE MOTHER AND FATHER, HOW WOULD YOU CUT THE CAKE SO

THAT EACH ONE HAS AS MUCH?

2. After the child has demonstrated one way, say: SHOW ME ANOTHER WAY TO CUT THE CAKE SO THAT EACH PERSON HAS AS MUCH.

Boxed-Candy Problem

1. Say: THESE BOXES OF CANDY ARE TO BE SHARED AMONG THESE CHILDREN. (Place the dolls on the table.)
2. Ask: CAN YOU GIVE THESE BOXES OF CANDY TO THE CHILDREN SO THAT THEY'LL EACH HAVE AS MUCH?
3. When the sharing is completed, ask: HOW MANY BOXES OF CANDY DOES THIS PERSON HAVE (indicating one doll)?

Chocolate Bar Problem

1. Say: THIS CHOCOLATE BAR IS TO BE SHARED AMONG THESE CHILDREN. (Place the dolls on the table.)
2. Ask: CAN YOU SHARE THE CHOCOLATE BAR AMONG THE CHILDREN SO THAT THEY'LL EACH HAVE AS MUCH?
3. Ask: HOW MUCH CHOCOLATE BAR DOES THIS PERSON HAVE?

The length of the clinical sessions varied from 20 minutes for the kindergarten children to 1 hour and 10 minutes for the older children. Usually, a five minute break was given after the child had worked through the Cookie Problem so this was a time for us to enjoy a cookie while we talked about matters of interest to the child.

A total of 19 children were interviewed twice, some to complete the tasks and others for further questioning.

Usually, three clinical sessions were conducted per day, two in the morning and one during the first hour of the afternoon school session. The data were collected over a period of six weeks.

H. Data Analysis

The sampling procedures described in a previous section (p.26) and the process of analysis presented here are based on a method of research designed to generate theory from data. Glaser and Strauss (1967) refer to this process as theoretical sampling and define it thus:

> Theoretical sampling is the process of data collection for generating theory whereby the analyst jointly collects, codes, and analyses his data and decides what to collect next and where to find them, in order to develop his theory as it emerges. This process of data collection is controlled by the emerging theory, whether substantive or formal
>
> (p.45).

Accordingly, a three-stage analysis scheme designed to maximize discovery was central to the purpose of the study.

1. Immersion Stage

The first stage of the analysis was an integral part of the clinical session. The stage was labelled "immersion" as it designated a continuous analysing process characterized by an intense focus on what the child said and did at each moment of the interaction. This ongoing analysis provided direction for immediate probing and for sampling behaviors within problems and problems within persons.

In being dynamically involved in the interaction, I was alert to a child's moment of frustration, distractedness, or manifestation of fatigue; an awareness which led, in a few instances, to the termination of a clinical session for that day.

In addition to the mechanical recording (video and audio) of the entire clinical interactions, distinctive behaviors observed and impressions made were recorded on a form during the session. A sample form is included in Appendix B.

2. Reflection Stage

Each evening, time was devoted to reflecting upon the day's interactions--an experience thought to be a vehicle to bring about, if at all possible, a breakthrough in awareness of patterns of behavior and their meaning to rational number development. Some video tapes were replayed; some of the taped sessions were transcribed and analysed; and, systematic written summaries were constructed. Weekends were a time to absorb the experiences of the week; a time for additional transcribing and re-viewing of tapes; and, often, a time for discussion with my advisor.

A journal was kept of insights gained and inferences made. The reflective process provided direction for deciding who to interview (an older or younger child; the same child) and what kind of questioning (the same nature but to a greater depth; a different nature) to pursue next.

3. Documentation Stage

Stage three of the analysis was conducted subsequent to the data collection and after all the audio tapes had been transcribed. This last stage comprised a systematic examination of the video tapes, the transcriptions, and the interpretations made during the first two stages of the analysis process. Moreover, charts, lists, and tables were constructed for the purpose of verifying the interpretations made and for gaining further insights. Extensive discussion with my advisor helped to formulate and systematize the discoveries.

Understandings and conceptions regarding the nature of partitioning behaviors are presented in chapter IV.

Although selected clinical session recordings were viewed by supervisory committee members as a verification measure, the data interpretations are those of the author. The process of theoretical sampling is demanding and of necessity the results are reflective of a researcher's personality, mental powers, and theoretical sensitivity. This is in accord with Glaser & Strauss (1967) who state that discovering grounded theory demands a certain degree of theoretical sensitivity which increases or deepens with experience over time. Furthermore, it involves the researcher's "personal and temperamental bent" and the "...ability to have theoretical insight into his area of research, combined with an ability to make something of his insights" (p.46).

The experience of data collection and simultaneous analysis resulted in insightful gains into children's partitioning behaviors which led to or form part of the final interpretations. The new knowledge emerged from the total experience and is "grounded in the data".

IV. PRESENTATION OF THE DATA

The purpose of the present study was to gain insights into the construction of the rational number concept in children. Understanding of the partitioning behaviors of young children was sought through observation and questioning techniques while children were engaged in solving partitioning problems by manipulating given materials.

Five tasks were used in the study. One, the Cake Problem, proved to be highly effective in enabling children to demonstrate partitioning capabilities and techniques. While interacting with children engaged in partitioning the selected regions, foundational elements of the partitioning mechanism were uncovered. Therefore, the descriptive data and major findings relate specifically to the Cake Problem.

A sample transcription of an interaction session with each problem is included in Appendix C.

In this chapter, perceptions attained and conceptions formulated regarding the capabilities of young children to construct partitions on circular and rectangular regions are presented and documented.

Selected transcripts are presented at the outset to enable the reader to obtain a feeling for the data. The transcripts describe interactions conducted near the end of the data collection period. As such, they show the researcher probing for confirmation of emergent insights; insights which the reader may "grasp" before reading the formalized account given later.

A. Selected Transcripts

Note: A diameter stick is a stick as long as a diameter of the cake and used to partition the region by a 'cut' through the center.

A radius stick refers to a stick shorter than a diameter of the cake and used to designate a 'cut' from the center of the region to the perimeter.

Bro, (8;5)

Bro (8 years, 5 months) had successively partitioned the giant cookie in half and fourths but was unable to get thirds or fifths. Presented with a rectangular cake, she partitioned it in half, thirds, fourths, fifths, sixths, eighths, and sixteenths. She was then given a round cake and was asked to share it among three children. Her first attempt was with two diameter sticks in the form of an "X"; then she removed one stick. Stating, THIS IS HARD, she proceeded to place two additional sticks ending with five pieces. THAT? "For three people? How many pieces do you have there?" FIVE. She moved the sticks and formed six pieces. "Now?" SIX. "Share it for six people." She adjusted the sticks by rotation moves and made even shares. "Suppose six more children arrived, what would you do?" She added three diameter sticks, bisecting each piece as she placed the sticks on the cake. "Fair share?" EMHEM. "What would you call this piece?" QUARTER. "How many quarters in that cake?" TWELVE. "Could you call it (one share) another way?" SIXTH. TWELFTH. While removing the sticks, I stated: "You have six pieces. We'd like to have five pieces." She made several unsuccessful attempts beginning either with two perpendicular diameter lines or vertical parallel lines. "What are the numbers that you're having difficulty with? That you find hard?" THREE AND FIVE. "Why?" CAUSE IT'S KIND OF HARD TO MAKE IT. "Why is it? You can make four and six and eight; you've made twelve and sixteen. Why are these numbers hard?" CAUSE IT'S THE LITTLE ONES. "The little ones are hard? Is two hard?" NO. "Four?" NO. "Six?" NO. "So, what's hard about three and five?" WELL, IF YOU TRY AND MAKE

SOMETHING OUT OF 'EM, YOU ALREADY MADE THEM. "What do you mean by that?" WELL, YOU TRY AND MAKE FIVE, BUT THEN YOU GET SIX. "Why is it that you get six when you try to make five?" I DON'T KNOW. IT'S KIND OF HARD. "I'd like to know why it's hard." CAUSE I NEVER DID FIVE AND THREE BEFORE. "Start again with two, and four, and six. Start there and try and show me why three and five are hard. You tell me everything that you're thinking as you're making those. How do you plan a four and how do you plan a six?" She partitioned the cake in fourths. "Tell me what you're thinking as you're making those." I'M THINKING THAT IF YOU GO LIKE THIS, FOUR COULD GO LIKE THIS BUT IF YOU PUT ANOTHER ONE LIKE THIS IT CAN MAKE EIGHT. "Alright, I'm glad you're telling me that. Now, could you tell me something more?" LIKE IF YOU MAKE THREE LIKE THIS (a horizontal diameter cut and two vertical radius cuts) YOU'D GET FOUR, SO, LIKE, IF YOU GO LIKE THIS (moving the sticks) YOU STILL GET FOUR. IF YOU GO LIKE THIS (one horizontal diameter cut), IT'S TWO HALVES. "Now, could you begin differently, though? Each time, there, you're beginning with this (diameter-cut)." I CAN BEGIN WITH A NEW LITTLE ONE LIKE THIS. "Alright, and keep on to make three." She made several attempts using diameter sticks, beginning with a halving cut. "Maybe it's how you begin. Could that be it?" EMHEM. She tried again and ended with two vertical parallel lines. "Are those fair?" EMHEM. "To me, this seems to be a little bit smaller than this....Try and begin with three pieces another way." She made several additional attempts and ended with a half piece and two quarter pieces. "Begin with three. You can see two pieces there, can't you? You can see four. Now, see three. Show me four with those sticks (radius) you have...." BUT I NEED A BIG ONE. "You could take another one of those." She did and partitioned the cake in fourths. "Now, you look at that, and say to yourself, I want three, not four. What can you do?..." She first removed two sticks then replaced one, and finally with rotation moves, succeeded in getting thirds. "Now, why was that difficult? Tell me that." I FORGOT HOW IT WAS DONE. "Have you done that before?" EMHEM. IN MATH LAST YEAR. "Did you do that last year? Oh, so, why is that difficult to do?" CAUSE I NEVER DID MATH FOR A LONG TIME. WE NEVER DID THIS IN MATH YET. "Now, if I ask you to do five, is that going to be difficult?" She added one stick, then rotated some, and added one more without hesitation. She went on to partition the cake in sevenths.

Jou (8;5)

Jou was asked to partition the circular cake in three fair shares and he did. "Why is it that for three, you didn't pick this (diameter stick) up?" CAUSE IF YOU PICKED THIS UP, IT'D BE GOING LIKE THAT (halving), THEN YOU'D HAVE TO PUT THESE ON (two radius sticks) AND IT WOULDN'T BE EVEN. SO YOU HAVE TO TAKE THREE SMALL ONES AND PUT THEM ON LIKE THAT. "I wonder why that's so?" CAUSE IF YOU USE THESE (diameter sticks), IT'S GOING TO BE ALMOST IMPOSSIBLE CAUSE IF YOU DO, YOU'D HAVE TO GET ANOTHER AND PUT IT LIKE THAT AND THEN YOU'D GET FOUR, AND THEN YOU'D NEVER GET THREE. "So, why doesn't three work with that?" CAUSE YOU COULDN'T GET THREE WITH THAT CAUSE IF YOU USE THOSE, YOU WOULD KEEP GETTING MIXED UP. YOU'D SKIP THREE IN EVERY...(He didn't complete his statement.) From three shares, Jou went on to partition the shape in sixths by halving each piece, then twelfths, and predicted the next number of shares would be twenty-four and then forty-eight. He was then asked: "What about five? Why is five not coming out?" (He had been saying the numbers he got: 2, 4, 8, 16; 3, 6, 12, 24.) CAUSE EVERY TIME YOU GO FROM THREE, YOU HAVE TO GO TO SIX CAUSE YOU ADDED THREE MORE TO MAKE IT EQUAL PARTS. AND FROM TWO TO FOUR AND IT WOULDN'T GO UP TO FIVE, ALSO IT WOULDN'T BE EQUAL. "So, the only way to get five is how?" USING THE SMALL ONES (sticks)...."Is five an easy one (number)?" SORT OF EASY AND SORT OF DIFFICULT. "What makes it difficult?" CAUSE WHEN YOU START, YOU THINK IT WOULD BE SMALL PARTS BUT WHEN IT ENDS UP YOU HAVE TO HAVE BIG ONES....He then partitioned the cake in ten pieces and then twenty, and predicted the next numbers would be forty and eighty. He was next asked to partition the cake in seven pieces. He picked up the small sticks and proceeded to make seven parts. "Is there another way to get seven besides one by one? Could you plan it?" TWO, FOUR, SIX. NO. LIKE YOU HAVE SEVEN PAIRS OF SOCKS. YOU'D HAVE ODD, YOU HAVE ONE MORE. YOU HAVE TWO THEN YOU HAVE FOUR. THAT'S KNOWN AS TWO PAIRS. THEN SIX IS THREE, AND THEN SEVEN. THAT'S THREE PAIRS AND A HALF. SO I DON'T THINK I COULD. "Do you know a name to call the number seven? What kind of a number is seven?" A ODD ONE. "Why do you call it odd?" CAUSE IT'S, IT'S LIKE HAVING HALF A PAIR OF SOMETHING. "What would you call eight?" EIGHT IS LIKE A PAIR. "Another name?" I FORGOT IT. I DONE IT IN GRADE ONE. YOU HAD THE ODD NUMBERS AND THE SOMETHING NUMBERS.

Hof (8;10)

Hof was at first unable to partition the giant cookie in thirds but he succeeded in attaining fifths. Following this achievement, partitioning in thirds was no longer a challenge. He was presented with a rectangular cake and he partitioned it in half and then in fourths using vertical, horizontal, and diagonal lines to obtain qualitatively different partitions. He did not, however, use three parallel lines to obtain fourths.

When asked to share the rectangular cake for groups of three and five people respectively, Hof was perplexed. His procedural moves were modeled on those performed for thirds and fifths on the giant cookie, that is, the use of radius sticks radiating from the center of the region. His thinking appeared to be unaffected by the particular structural properties of the two shapes. Varied sized rectangular cakes were presented to him in an effort to break his pattern of moves. He partitioned the large cake in thirds using horizontal parallel lines but he did not employ this strategy in his attempts to obtain fifths. He was then asked to share the large cake for seven people but he did not succeed. The cake was changed once again. "Could you share this cake (narrow) for seven people?" THAT MIGHT BE MORE EASIER BUT I'LL TRY. He made several unsuccessful attempts. THIS ONE'S AN ODD NUMBER. "Pardon?" I SAID, I CAN DO EVEN NUMBERS BUT IT'S HARD TO DO ODDS. "What do you mean by an odd number?" LIKE TWO, FOUR, SIX, EIGHT. "What are those?" THEY'RE EVEN NUMBERS. "I see. And which ones do you find it difficult to get pieces for?" ODD. "Why?" CAUSE IT GETS TOO HARD, LIKE, I, THREE PIECES IS HARD, FIVE, AND SEVEN. "Okay, suppose you had seven people and you wanted to give each a piece of cake." EMM, THIS IS HARD THOUGH. He made further attempts but was not successful.

During a second clinical session, Hof was presented with a round cake and, at my request, proceeded to partition it in halves, thirds, fourths, and fifths. Observing the cake partitioned in fifths, he was asked: "What is each part called?" A QUARTER. "How many quarters do we have there?" FIVE. When requested to share the cake among six people, he picked up two diameter sticks and placed them on the cake saying: THERE'S FOUR. I'D BETTER TRY IT A DIFFERENT WAY. He removed the sticks and picked up some radius sticks while saying to himself: SIX,--EMM, THAT'S NOT GONNA BE EVEN. He positioned them correctly on the cake. THERE. "What is each part called?" A QUARTER. To partition the round cake in sevenths proved to be a challenging task for Hof. He first made six partitions and then removed all the sticks. LET ME TRY THIS AGAIN. He used diameter sticks and obtained eight sections. He changed again to the radius sticks and produced six partitions but, as he placed the last stick on the cake, he stated: THAT WOULD BE EIGHT and subsequently removed all the sticks. I CAN'T THINK OF A WAY. He tried again, beginning first with the radius sticks and made fourths; then he halved two of the sections; next, he removed the last two sticks he had placed on the cake and used them to bisect two different sections; lastly, he removed all the sticks. I CAN'T GET IT. IT'S JUST TOO HARD. He picked up three diameter sticks and partitioned the cake in six pieces then removed them. I JUST CAN'T GET IT. "Why?" IT'S JUST TOO HARD. "Can you get eight?" YES. "Nine?" MAYBE. He used two diameter sticks to make quarters, placed another diameter stick, and then another to obtain eighths but he stated: THERE'S TEN. He removed the sticks and proceeded to place seven radius sticks in turn on the cake producing seven pieces and said: THERE'S EIGHT THERE. I'LL TRY BUT I DON'T KNOW HOW. He again placed seven radius sticks on the cake, counted the pieces, and exclaimed: NOW, I'VE MADE SEVEN! "Are they a fair share?" NO. "Make them a fair share." He did. "That's the one you had difficulty with." YES...."Now try to get nine." Using radius sticks, he first produced fourths, then eighths. EIGHT! He removed the sticks and began anew with three diameter sticks making six partitions (not congruent). SIX! He added two radius sticks. IF I DO THAT, THAT'S TEN. OH, BOY! He made several other attempts, usually ending with eight partitions. He reverted to parallel lines (two vertical and two horizontal) producing nine uneven shares. "Is that a fair share?" NO. "Why not? "CAUSE THERE'S A ROUND THING HERE, BUT IT'S ROUNDED, THERE'S A SQUARE HERE (in the center), BUT IT'S A QUARTER PART HERE. "So, that's not the way to get nine." He made a new effort with the radius sticks and this time succeeded. NOW I HAVE TO MAKE THEM EVEN. "Alright, you've just made what?" NINE. "And it took you a while to get that." YA. "Why was that difficult?" CAUSE I KEPT PUTTING THEM TOO WIDE. "That's one good reason. Can you tell me some other reason why nine would be hard?" CAUSE NINE IS AN ODD NUMBER. Another very good reason. (Pause.) What does that mean when you say it's an odd number?" IT MEANS LIKE TWO, FOUR, SIX, EIGHT, TEN,--THEY'RE EASY TO GET BUT THE ODD NUMBERS ARE HARD. "What do you call the easy numbers?" EVEN. "What does that mean, a number that's even?" IT MEANS LIKE, EHHH--IF YOU'RE COUNTIN' BY TWOS, THEY'RE EVEN NUMBERS. IF YOU COUNT BY ONES, LIKE, I SHOULD SAY BY THREES--ONE, THREE, LIKE THAT, THEY'RE THE ODD NUMBERS. "What's the one after three?" ONE, THREE, FIVE, SEVEN, NINE. "How are you counting there?" BY ONES, NO--THAT'S HARD. YOU'RE COUNTING BY ODD NUMBERS, BUT I CAN'T THINK OF WHAT NUMBER YOU'RE COUNTING BY. "Okay. Is there another reason why nine was difficult?" I CAN'T THINK OF ONE. "What have you discovered now, then?" THAT IF YOU PUT THEM IN A BIT MORE SMALLER, YOU CAN MAKE NINE. "Okay, what would be the next number that might be difficult to get?" ELEVEN. "Would you want to try that?" OKAY, I'LL TRY. He began placing radius sticks on the cake (round) and stopped when he had ten. I'LL MOVE THEM CLOSER. IF I PUT THAT ONE THERE, WHAT WOULD THAT BE? (He was talking to himself.) He counted them. There were eleven. "Fair share?" YES. "In order to make it a fair share, what must you be careful to do as you're placing those sticks there?" CUT THEM EVENLY. "Yes. Do you place those sticks anywhere?" NO. "Where? What are you thinking of as you're placing those sticks there?" MAKING SURE THAT THEY ALL GO IN THE CENTER.

I wanted to find out if Hof could produce a number of partitions by trisecting sections, so he was asked to share the round cake for three people. I DID THAT BEFORE. "Why did you immediately pick up those sticks (radius) and not these (diameter)?" CAUSE IT WOULD BE TOO LONG AND IT WOULD GO ALL THE WAY ACROSS. THESE STICKS ARE EASIER TO WORK WITH. "Okay, for certain numbers." YA. "Let's try this one (diameter stick). Is it possible to get three beginning with that one?" NO, WE WOULD JUST GET TWO. IF YOU ADDED ON THAT ONE YOU WOULD GET FOUR. "And so you don't think you can get three using those sticks?" NO. "Let's put your three here again. Now, once you have three this way, what is the next easy number to get with that?" NINE. "Alright, without moving those sticks, do it." He bisected each section. "What do you have?" SIX. "That's a rather nice way to get six, isn't it?" YAP...."You said that you could get nine from that (thirds). I'd like you to try to get nine without removing these sticks. Just look at the cake and think of what you have to do." He trisected one section. "Now, why did you decide to do that? Just stop there." CAUSE IF YOU PUT THEM SMALL YOU MIGHT BE ABLE TO FIT NINE IN. "Okay, so you're not sure you're going to, though, eh?" I MIGHT. "You have to move some sticks, do you think?" NO. "How many small pieces do you have there now?" THREE. "And look at what's left. Do you think that you are going to arrive at nine?" OKAY. He counted to nine while staring at the cake. YES. "Are you sure?" YES. "Why?" CAUSE IF I KEEP ON PUTTING THEM LIKE THIS, THEN I--I WILL! "How do you know? Tell me." CAUSE IN MY HEAD, I DID THAT. (He demonstrated trisections.) "And what does that get you?" IF YOU DO THAT, THERE'S ONE, THERE'S THREE, FOUR, FIVE, SIX, SEVEN, EIGHT, NINE! "Very good. So, what are you doing to each one of these pieces?" I'M PUTTING THEM INTO SMALL QUARTERS. "Yes, and how many pieces?" NINE. "In each one of them? How many pieces will you get in each space?" EMM, THREE. "Three more which gives you?" SIX. "And there?" THAT WOULD GIVE YOU THREE AND THEN IF YOU ADD ON IT WOULD BE NINE. "Good. Do it now. I know that you know how to do it." ..."Now, what about five? What's the best way to plan a five?" PUT IT IN THE MIDDLE.... "Just think for a minute. Is there another way that you could plan for five? Could you get five from a smaller number? Like from two you get four and from four you get eight." He partitioned the cake in halves, then fourths, then fifths. IT'S THE ONLY WAY I CAN THINK OF. "Why is it the only way?" CAUSE YOU HAVE TO PUT SOME CROOKED, LIKE IF YOU PUT A LONG ONE THERE, IT WOULDN'T BE FIVE. "It would not come out to be five, eh?" YA. "Having done it that way, what's your next number to get?" TEN. "And after ten?" TWENTY. "Okay, good! Is there another number you could get from five?" BESIDES TWENTY? "Yes." (Pause.) "I'm thinking of fifteen. Tell me how you'd get fifteen from five." I'D HAVE TO SHOW YOU BY USING THE STICKS. "Show me by dividing one piece." He demonstrated bisections of each piece. I DON'T KNOW HOW YOU COULD GET FIFTEEN. "I think you do." UNLESS YOU MOVE ALL OF THESE OVER. "No, we're going to leave them there." OKAY. "If you do it that way (bisecting each piece), what happens? How many pieces do you get?" TEN. "So, that doesn't help you to get fifteen, does it?" NO. "So, there's got to be another way." IF YOU DID IT LIKE THAT. He indicated a trisection of one piece. "Will that work?" YES. "Are you sure? Prove it to me without even doing it." OKAY. He counted aloud, pointing to imaginary parts of the cake. ONE, TWO, THREE; 4, 5, 6; 7, 8, 9; 10, 11, 12; 13, 14, 15. He laughed. This was evidently a new experience for Hof and he was demonstrably pleased with his accomplishment.

Doz (9;8)

Doz was attempting to section the circular cake in thirds. "For which number is it difficult to do?" FOR THREE.. "Is it difficult to do it for four?" NO. "What about five?" YES, HARD. "What about six?" EASY. "Seven?" MAYBE HARD. "Eight?" EASY. "Why is it easy to do it for two, four, six, and eight?" BECAUSE WHEN YOU TRY TO DO THREE, IT DOESN'T WORK. WHEN YOU TRY TO DO SEVEN, IT'S HARD BECAUSE IT MIGHT NOT WORK. "What do you mean when you say it doesn't work?" WELL, IT WON'T FIT. "Alright, can you show me besides telling me?" She proceeded to partition the cake in halves, fourths, and sixths. "Now tell me why it's difficult to make it for three." WELL, IF YOU'LL PUT IT LIKE THAT (cutting it in half) AND THEN TRY TO MAKE THREE, YOU'LL HAVE TO PUT IT LIKE THAT (a radius stick perpendicular to the halving cut), IT WON'T BE EVEN. "For three, then, can you start that way?" YES. "Alright, do it." She tried. "To make it a fair share for three, can you start that way?" NO, IT WON'T WORK. "Alright, so then, it seems that if it doesn't work, you have to start another way, don't you think? What would be another way to start for three?" She placed two sticks in a vertical parallel position. "Yes, that's one way, and you have three pieces. Now, are they a fair share?" NO. "Why?" BECAUSE THIS IS THE CORNER AND THIS IS THE MIDDLE. "So, there must be another way to begin for three to make it a fair share." She placed two radius sticks on the cake. "How many pieces there?" TWO. She moved one to form a right angle with the other stick and placed a third stick on the cake. With rotation moves, she constructed what looked like congruent pieces. "Alright, what did you do there to get three?" WELL, FIRST I PUT THIS HERE AND THEN I PUT THIS ONE, I CUT IT. "Do it for four." She did. "Easy or hard?" EASIER. "Why?" WELL, FOR THREE, YOU PUT THAT (radius stick) FIRST, THEN GO IN THAT WAY AND YOU HAVE TO CUT IT LIKE THAT. "So that's for four. Our next number is what?" FIVE. "Do it for five." She placed four vertical parallel lines on the cake. "Is that a fair share?" NO. "Why not?" BECAUSE THE CORNER PIECES ARE BIGGER. "There must be another way to do it for five." She picked up a diameter stick but changed it and began to section the cake using radius sticks and easily obtained fifths. "Now, how did you begin that one?" WELL, I HAD ONE OF THESE (diameter stick) ACROSS LIKE THAT (slanted) AND THEN I TOOK IT OFF. "Why?" BECAUSE IT WOULDN'T WORK IF I DID THAT. THEN I TOOK A SMALL ONE AND THEN I PUT IT HERE AND I PUT TWO MORE HERE AND TWO MORE HERE....She partitioned the cake in sixths (beginning with fourths, adding a diameter stick followed by rotation moves) and was asked to do it for seven people. She called it a "hard one" because, as she stated, IT MIGHT NOT WORK. "How are you going to try it?" WITH THE LITTLE ONES. She did it easily....Later, she was asked why three, five, and seven are hard to make and she answered: WELL, IT WON'T GO IN EVEN PIECES.

B. The Findings

A systematic analysis of the partitioning behaviors of young children revealed distinctive levels of capabilities. These levels, together with knowledge of elemental conceptual structures necessary for constructing unit fractions, have led to the formulation of a theory for the development of the partitioning mechanism.

The theory emerged from the data; therefore, in expounding the theory, the major findings are presented in a general mode.

In the following section, foundational elements undergirding the partitioning construct are presented. Next, the partitioning behaviors of young children are detailed within seven categories. The geometric figures used in the study are discussed under two headings: first, the effect of structural properties on the construction of partitions and second, the difficulty level of the selected figures. Lastly, the theorized five stages of partitioning capabilities which a child must attain before the partitioning mechanism is mastered are defined.

The Partitioning Construct: Foundational Elements

The selected transcripts cited in the preceding section provided, in part, information which led to the identification of foundational elements of the partitioning construct. Briefly stated, it is proposed that the conceptual structures necessary to construct unit fractions have their genesis in basic number theory and transformation geometry.

In particular, ideas of even and odd, prime and composite, factors and multiples play key roles. Moreover, elements of geometry interplay with number theory concepts as the child endeavors to master partitioning mechanisms. Notions of transformation geometry such as translations, rotations, symmetry, similarity, congruency, and the structural properties of the shapes all play a significant role.

The role of these foundational elements in partitioning rectangular and circular regions is the focus of the remainder of this chapter.

Partitioning Behaviors: Categories and Description

The partitioning behaviors of children demonstrate a progression through several levels of capabilities. These are described below under seven categories.

Early Partitioning Behaviors

Early attempts at subdividing rectangular or circular regions in halves, thirds, fourths, or fifths produce a number of parts greater than the number required, uneven shares, or only a portion of the region is partitioned. Samples of these partitioning attempts are depicted in Figure 4.

Early Partitioning Behaviors: Circular and Rectangular Regions

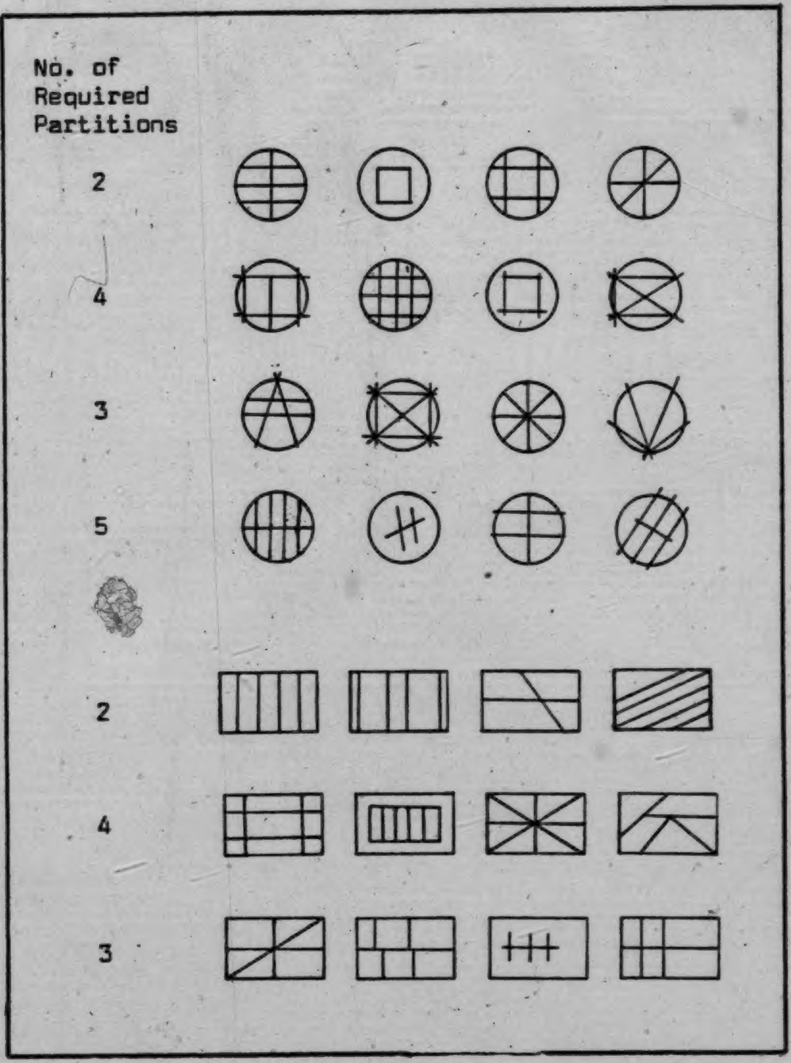

No. of Required Partitions
2
4
3
5
2
4
3

Figure 4

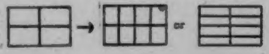
The Halving Mechanism

A child first learns the 'halving mechanism', that is, he learns how to partition in half. The young child's partitioning activities are for some time dominated and controlled by the halving mechanism; that is, by a line or 'cut' through the middle of the region. This is true whether the shape is rectangular or circular.

Some children who can successfully partition in halves and fourths produce incorrect partitions on successive trials. Others begin with uneven shares or too many shares, but, when requested to share the cake so that there are only two or four parts with 'all the cake used up', proceed to partition correctly in halves or fourths. Such erroneous attempts demonstrate that these children, although at times successful, have not attained mastery of the halving mechanism and do not realize that evenness is a necessary quality for halving.

The child who has learned the halving mechanism is able to make the partitions he or she may need to make; for example, halves and fourths. Both rectangular and circular shapes are at first partitioned in half by a vertical or horizontal line while a slanted or diagonal line is employed at a later time.

Partitioning in fourths is first achieved by a halving line (vertical or horizontal) and its perpendicular bisector. Again, the use of diagonal lines is somewhat delayed. Seldom is a rectangular shape partitioned in fourths by three parallel lines prior to partitioning the shape in thirds. The shape of the figure may have something to do with the type of sectioning a child performs. For example, a long narrow rectangular shape may more readily be sectioned with three parallel lines than by two diagonals. In this investigation, 'cakes' with dimensions 10 cm by 15 cm and 15 cm by 20 cm were most frequently employed for sectioning in fourths.

In time, a child uses the halving mechanism in an algorithmic manner, that is, he or she can perform successive dichotomies. Thus, from partitioning in halves and fourths, a child quite easily proceeds to partition in eighths and even sixteenths if his or her counting ability and knowledge of addition are adequate. For example, when a rectangular region has been evenly partitioned by two perpendicular lines and the child is asked to make four additional partitions, young children who have learned the halving algorithm unhesitatingly proceed to bisect the two half sections by adding either two vertical or two horizontal lines, thus producing eighths []. In a like manner, children are able to obtain sixteenths. Partitioning a circular shape in eighths from fourths is performed with similar ease. This adeptness at doubling the number of partitions is a manifestation of mastery of the halving mechanism.

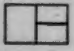
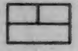
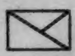
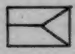
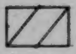
Rectangular Figure: Thirds

The halving mechanism is a tool which serves the child well for solving many of his or her partitioning problems. However, when the child is faced with the task of partitioning in thirds, this acquired tool becomes somewhat inadequate, but not totally so.

For example, if the child is asked to partition a rectangular region in thirds, a vertical line from top to bottom as in halving can be of service. The only change is that the line cannot be positioned through the middle of the region. The child must come to realize that for thirds, two parallel lines are necessary. It may appear to be a trivial move from halving a rectangular shape to partitioning it in thirds, but for the young child, it is a major accomplishment.

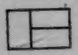
Before thirds are obtained, a young child partitions a rectangular shape in three parts in different ways. When the child's thinking is dominated by the halving mechanism, a first attempt at thirds is to make three parallel lines thus producing four pieces. For the young child, the rule is: Make as many lines as the number of required shares. Lon (5-6) voiced this rule. When asked to share the giant cookie with four children, he stated: FOUR OF THEM, SO I HAVE TO PUT FOUR [sticks]. He placed all the sticks in a vertical position. Later, when presented with three dolls, he said: I WOULD PUT THREE ON 'EM.

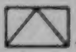
Other early attempts at partitioning in thirds are:

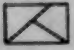
1. Partitioning in half by a vertical or horizontal line and halving one of the sections with a perpendicular bisecting line [ ].
2. Partitioning in half by a diagonal line and subdividing one of the pieces by a semi-diagonal line [].
3. Partitioning by working from the center of the region and making three lines: a horizontal line to the midpoint of one side (width) and two oblique lines to each vertex of the opposite side [].
4. Partitioning by making two parallel oblique lines [].

Such products demonstrate that the child's thinking is dominated by the halving mechanism. In time, the child comes to realize the unevenness or unfairness of such partitions and subsequently attempts to make fair shares.

Examples of such efforts are:

1. Instead of partitioning in half, the 'halving stick' is transposed away from the middle line making two unequal pieces, and the larger section is halved with a perpendicular bisecting line [].
2. Partitioning by two oblique lines from the base vertices and intersecting at the midpoint of the

opposite side [].

3. Partitioning by an oblique line from a base angle to a point (around the middle) of the opposite side and by dividing the larger piece in two by a line from the other base angle to a point on the first line [].

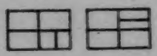
It must be noted that some of these partitions could be even shares.

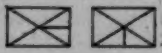
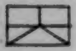
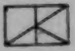
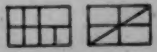
As previously stated, partitioning a rectangular shape in fourths using three parallel lines (vertical or horizontal) is seldom used prior to the child's capability of partitioning in thirds although parallel lines are used in early unsuccessful attempts at thirds, fourths, and fifths. In such instances, the sticks are either not properly positioned or they are not the correct number of lines for the task at hand.

Rectangular Figure: Fifths

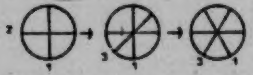
At the time when uneven partitions in thirds are made due to the dominance of the halving mechanism, a child produces homologous partitions for fifths. Early attempts at fifths are generally built on from fourths.

Examples are:

1. Partitioning in fourths by two perpendicular bisecting lines and by halving one of the quarter pieces by a vertical or horizontal line [].

2. Partitioning in fourths by two diagonal lines and halving one of the quarter sections [].
3. Partitioning in half by a horizontal line and then bisecting the upper section by a vertical line and trisecting the other by two oblique lines from the base angles to the central point of the region [].
4. Partitioning in four pieces by a vertical halving line and one diagonal line and then subdividing one of the larger pieces by an oblique line from the central point to the opposite vertex [].
5. Partitioning in fourths by two perpendicular bisecting lines and then making additional lines to produce more than five pieces [].

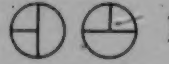
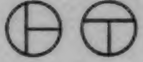
Circular Figure: Thirds

Partitioning a circular shape in thirds presents a greater challenge than partitioning a rectangular region. As previously stated, the halving mechanism enables the child to easily partition in fourths, eighths, and sixteenths. It also helps the child to partition a circular shape in sixths, tenths, and twelfths. These latter partitions are achieved by rotation moves. For example, to obtain sixths, the child first partitions in fourths, then adds another line producing six sections. Lastly, a rotational movement of sticks 1 and 3 produces even shares in sixths [

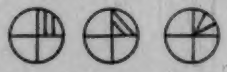
]. Tenths can be constructed in a similar manner by building on from eighths. Twelfths are easily attained from sixths by halving each section.

To partition a circular shape in thirds, however, is a non-trivial task for a young child. The halving tool is no longer adequate. When the child realizes that thirds cannot be attained by the halving mechanism, he or she begins a search for a new first move or line. This first move is, of course, a radius of the circle. Before the new tool is discovered, a young child partitions a circular region in thirds in various ways.

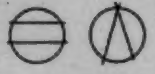
Building on or employing the vertical or horizontal line as in halving, some children partition in thirds by constructing three parallel lines thus obtaining four sections. Here again, the rule is "Three lines for three shares". When requested to allocate the shares, there is generally an expression of surprise at finding one piece 'left over'. Some children, after noticing the extra piece, proceed to remove one stick and translate the other two to form three shares with "all the cake used up".

In attempting to partition a circular shape in thirds, some children begin by halving and then proceed to bisect one of the half pieces []. When questioned about the size of the shares, some children will alter the partitions by translating the 'halving stick' to make the half piece smaller and the two quarter pieces larger, thus producing 'fairer shares' [].

Other early attempts at constructing thirds are:

1. Partitioning in fourths to produce three even shares and one piece left over.
2. Partitioning in fourths and then subdividing the extra piece into three pieces. This subdivision is produced in various ways: the lines may be perpendicular to a halving line; they may be slanted; or they may be radii of the circle [].

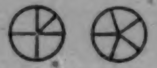
Sometimes, three sticks are placed on the fourth piece, thus producing four small sections.

The above methods are generally employed prior to partitioning the circle in three sections using only two parallel lines either vertical or horizontal, or with two lines intersecting at a point on the perimeter [].

When the child comes to realize the unevenness of such partitions, a first reaction is: "It can't be done" or "Three is hard." If pressed to make only three pieces with a statement like: "You can get two pieces and four pieces, now get three pieces", the child usually picks up one or several radius sticks and begins 'cutting' from the centre of the cake. At this level, the child knows that thirds cannot be obtained by beginning with a halving line. Some children who have attained this level, work through several trials by placing three or four radius sticks intersecting at right angles at the center and still claim "I can't do it." Others, after trial and error, move the sticks from a perpendicular intersection to correctly partition in thirds. Still others, place one stick on the cake and then 'see' where the other sticks go and, as if in one move, place the sticks in a correct position to obtain thirds. Having achieved this feat, a child can easily proceed to partition in sixths from thirds.

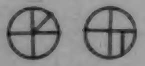
Circular Figure: Fifths

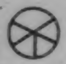
Attempts to partition a circular shape in fifths follows a similar process as thirds and does so concurrently.

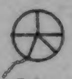
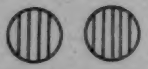
When radius sticks are employed, a child can achieve fifths from fourths by first halving one of the quarter sections and then by rotation moves, reposition the sticks to obtain five even shares [].

When a child's thinking is dominated by the halving mechanism, he or she has several techniques for partitioning a circular shape in fifths.

Examples are:

1. Partitioning in fourths by a diameter line and its perpendicular bisector and then halving one of the sections [].
2. Partitioning in fourths by two oblique

diametrical lines and then halving one of the sections. The two lines may or may not intersect at right angles. In the latter case, one of the obtuse angles is bisected [].

3. Partitioning in half by a vertical or horizontal cut and then bisecting one of the half sections and trisecting the other with lines originating at the center [].
4. Partitioning in five or six sections with four or five parallel chords positioned equidistant from each other [].

Cognizant of the unevenness of such partitions and having discovered the new tool for thirds, the child freely uses it to successfully partition a circular shape in five congruent sections. Thus, it can be said that a child achieves mastery of thirds and fifths and for that matter, sevenths, at the same time.

Some children are successful at partitioning a circular shape in five congruent parts prior to thirds. It appears that young children are not inclined to construct a piece which is larger than a fourth. Since one fifth is smaller than one fourth, a child readily begins to construct the pieces one by one for fifths. Frequently, the child ends up with five pieces before the whole cake is 'used up'. Here again, rotation moves change the uneven partitions into even shares with all the cake 'used up'.

Circular Figure: Ninths

Partitioning in ninths may be the last problematic number for the young child. Once the child has mastered thirds, fifths, and sevenths, ninths are obtained in a similar manner; that is, by using nine radius sticks and placing them successively on the cake. This counting algorithm is awkward and does not lend itself to producing even shares as many adjustments must be made before all the pieces 'look' congruent.

Since young children's experiences with subdivision of parts are almost exclusively a halving procedure, they do not think of partitioning in ninths by treating nine as a composite number; that is, as a multiple of three. However, if pressed to do so, it is probable that children who have mastered thirds would succeed at the task.

Hof (8;10) was asked to get ninths from thirds. He first bisected each section and realized this procedure would not enable him to get nine pieces. When told to "Just look at the cake and think of what you have to do", he proceeded to trisect each piece. Later, he succeeded at partitioning in fifteenths by trisecting fifths.

The Selected Figures

The selected figures, rectangular and circular, are discussed under two headings: structural variances and difficulty levels.

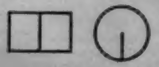
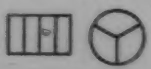
Structural Variances

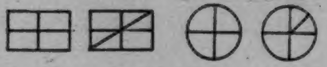
Rectangular and circular shaped objects were selected for the study because of the structural variances in the geometric figures. These varying properties proved to be stumbling blocks for some children causing them difficulty in choosing procedural moves.

A technique which was frequently employed was the use of parallel lines. While this strategy can produce congruent partitions on a rectangular figure, it is an awkward method for achieving equality of parts on a circular region.

Some children think that actions performed on the two kinds of regions yield identical results. The fact that this is not always the case is confusing to them.

Examples of differing products attained from the same operation are:

1. One line on a rectangular region produces two shares, whereas, on a circular region, one line (a radius) is not a partition. You do not get a 'share' from one radius cut [].
2. Three parallel lines on a rectangular region produce four pieces, whereas, on a circular region, three lines (radii) produce only three pieces [].
3. Adding a line to a rectangular region partitioned in fourths by intersecting lines produces two additional pieces, whereas, adding a line (radius) on a circular region produces only one additional

piece [].

Partitioning behaviors unaffected by such contrasting results indicate a conceptualization of the distinctive characteristics of the two geometic figures.

Difficulty Levels

More children were successful at partitioning a rectangular region in halves and fourths than in thirds and fifths.

All 43 respondents were asked to partition a rectangular region in halves and fourths; of these, 40 were successful at partitioning in both halves and fourths. The three children who were unsuccessful at partitioning in halves were also unsuccessful at partitioning in fourths.

Partitioning a rectangular region in thirds was attained by 25 of the 39 children who were presented with the problem, while 21 out of 35 children succeeded in partitioning the region in fifths.

Thirty-three children were requested to partition a rectangular region in both thirds and fifths. From this group, six were unsuccessful at both sized partitions; four were successful at partitioning in thirds only, while two were successful at partitioning in fifths and not in thirds.

Table 1 presents, by grade levels, the number of children who were successful and unsuccessful at partitioning in halves, thirds, fourths, and fifths on a

Table 1

Rectangular Region: Successful and Unsuccessful Partitioning by Grade

Grade	K		1		2		3		Total	
No. of Required Partitions	Yes[a]	No[b]	Yes	No	Yes	No	Yes	No	Yes	No
2	6	2	7	1	12	0	15	0	40	3
3	1	5	4	2	9	3	11	4	25	14
4	6	2	7	1	12	0	15	0	40	3
5	0	3	2	2	9	2	10	5	21	12

[a] "Yes" indicates a successful partition.

[b] "No" indicates an unsuccessful partition.

rectangular region.

On a circular region, more children were successful at attaining halves and fourths than thirds and fifths. This is similar to the results on a rectangular region.

Thirty-seven children were asked to partition a circular region in halves; of these, 33 were successful. All 43 respondents were requested to partition a circular region in fourths; of these, 37 were successful.

From the children who were not successful, only one failed both tasks. This child had also been unsuccessful at partitioning a rectangular region in halves and fourths.

Five children who were successful at halving a circular region were unsuccessful at partitioning it in fourths. Two children who attained fourths on a circular region were unsuccessful at attaining halves.

The circular region proved to be more challenging than the rectangular region for attaining thirds and fifths.

Twelve children from the 37 who were requested to partition a circular region in thirds were successful while ten out of 30 children attained fifths on the same shape.

Twenty-eight children were asked to partition a circular region in both thirds and fifths. From this group, 10 children were successful in attaining thirds and fifths while 17 were unsuccessful; only one child attained thirds and not fifths.

Table 2 presents, by grade levels, the number of respondents who were successful and unsuccessful at

Table 2

Circular Region: Successful and Unsuccessful Partitioning by Grade

Grade	K		1		2		3		Total	
No. of Required Partitions	Yes[a]	No[b]	Yes	No	Yes	No	Yes	No	Yes	No
2	6	2	8	0	9	2	10	0	33	4
3	0	7	1	3	2	9	9	6	12	25
4	4	4	7	1	11	1	15	0	37	6
5	0	4	0	3	2	6	8	7	10	20

[a] "Yes" indicates a successful partition.

[b] "No" indicates an unsuccessful partition.

attaining halves, thirds, fourths, and fifths on a circular region.

The percentage of successful partitions achieved was consistently as high or higher for the rectangular region than for the circular region except for one case; that is, 100 percent of the grade one children attained halves on a circular region while 88 percent of the children attained halves on the rectangular region.

The total sample was asked to partition a rectangular region in halves and fourths. Ninety-three percent of the children were successful at both tasks.

On a circular region, 89 percent of the children who were asked to partition in halves were successful while 86 percent of the total sample were successful at fourths.

Table 3 presents, by grade level, the percentage of successful partitions achieved at halves and fourths on rectangular and circular regions.

There is a greater discrepancy between the number of successful attempts at thirds and fifths on the two regions than at halves and fourths.

Sixty-four percent of the children who were asked to partition a rectangular region in thirds and fifths were successful while 32 and 33 percent were successful at thirds and fifths respectively on a circular region. Such discrepancies appear to indicate that a circular region is more challenging than a rectangular region for attaining odd numbers of partitions.

Table 3

Rectangular and Circular Regions: Percentage of Successful Partitions Achieved, Halves and Fourths, Grade Levels

	Rectangular Region		Circular Region	
Grade Levels	Halves	Fourths	Halves	Fourths
K	0.75[a]	0.75	0.75	0.50
1	0.88	0.88	1.00	0.88
2	1.00	1.00	0.82	0.92
3	1.00	1.00	1.00	1.00
Total	0.93	0.93	0.89	0.86

[a] To be read: 75 percent of the children who were asked to partition the region in half were successful.

Table 4 presents, by grade levels, the percentage of successful partitions achieved at thirds and fifths on rectangular and circular regions.

Stages of Partitioning Capabilities

An interpretation of the partitioning behaviors of the children reveals four distinct stages of capabilities in the development of the partitioning mechanism for rectangular and circular regions. Although not grounded in the data, a fifth stage is proposed as a logical next level. The stage is beyond the capabilities of the children who participated in the study, but it is presented as a last necessary stage for attaining mastery of the partitioning mechanism.

Stage I. The Sharing Stage

A child first learns to partition an object or a set of objects in the context of a social setting (Kieren, 1980 a). At an early age, a child is witness to situations where things are shared and hears the commonly used expressions such as, "Break (or cut) it in half" and "Here's half". Later on, the child participates in sharing activities and learns how to do the sharing and eventually uses the experiential language to describe the sharing action and the result of the sharing. But, as concerns number, this is rote learning. The child does not know the meaning of half in a quantitative sense nor are the necessary characteristics of evenness and one of two parts understood. This is evidenced

Table 4

Rectangular and Circular Regions: Percentage of Successful Partitions Achieved, Thirds and Fifths, Grade Levels

	Rectangular Region		Circular Region	
Grade Levels	Thirds	Fifths	Thirds	Fifths
K	0.16[a]	0.00	0.00	0.00
1	0.67	0.50	0.25	0.00
2	0.75	0.82	0.18	0.25
3	0.73	0.67	0.60	0.53
Total	0.64	0.64	0.32	0.33

[a] To be read: 16 percent of the children who were asked to partition the region in thirds were successful.

by the child's use of the term half in expressions like: BREAK IT IN HALF IN FOUR PIECES and SPLIT IT IN HALF IN THREE PIECES. Also, some children use the term half exclusively as an operation. An example is Lon (5;6) who reiterated that the cookie was broken in half but called each piece A BROKEN COOKIE. To the query, "What could you call that piece?", he responded: I DON'T REALLY KNOW.

During the sharing stage, the child has learned the halving mechanism, that is, a line through the middle of the region. The child uses this learned mechanism to partition both rectangular and circular regions. Although at this stage, a child succeeds at partitioning these regions in half and fourths, some attempts result in uneven shares, more parts than the number required, or the region is not all used up. During the sharing stage, partitioning continuous quantities simply means allocating "pieces" and an equal number, regardless of size, is considered a "fair share".

The partitioning behaviors of two children who are at the sharing stage are presented in Figure 5.

Stage II: The Halving Algorithm Stage

A second stage is marked by mastery of the halving mechanism. At this stage a child can successfully partition a rectangular or circular region not only in halves and fourths but also in eighths and sixteenths. In other words, the child can double the number of partitions or attain

The Sharing Stage
"Allocating pieces: more pieces than necessary."

The Sharing Stage
"Rule: 3 children, 3 cuts."

The Halving Algorithm Stage
Obtaining Eighths from Fourths

The Halving Algorithm Stage
Obtaining Sixteenths from Eighths

fractional parts whose denominate numbers are powers of two.

The child uses the acquired tool, that is, the halving mechanism, in an algorithmic manner; there is yet no notion of equality. Successive dichotomies are performed systematically so that each part is in turn

Partitioning Behaviors: Rio (5;0), Lon (5;6)

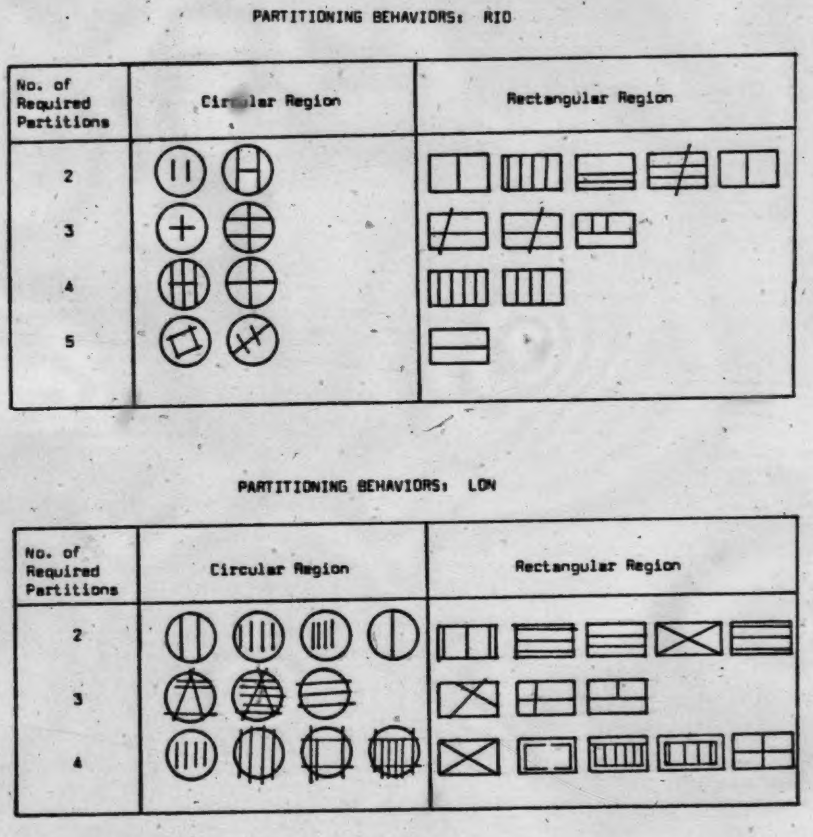

PARTITIONING BEHAVIORS: RIO
No. of Required Partitions
Circular Region
Rectangular Region
2
3
4
5
PARTITIONING BEHAVIORS: LON
No. of Required Partitions
Circular Region
Rectangular Region
2
3
4

Figure 5

bisected, thus doubling the previous number of partitions.

Examples of attempts at eighths and sixteenths are portrayed in Figure 6.

Partitioning Behaviors: Eighths and Sixteenths, Circular and Rectangular Regions

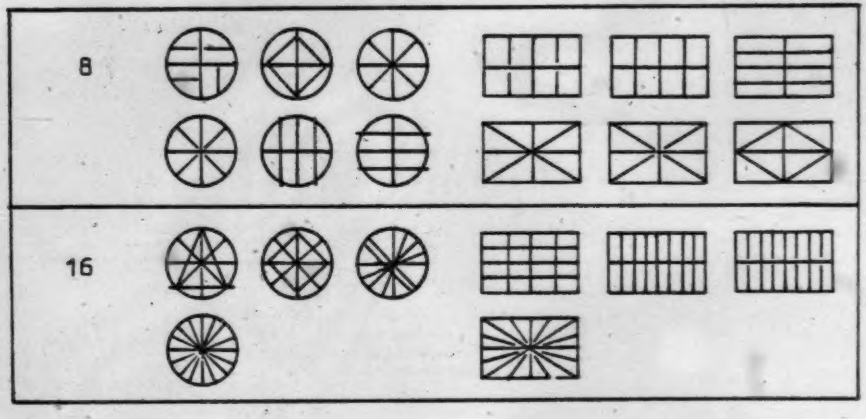
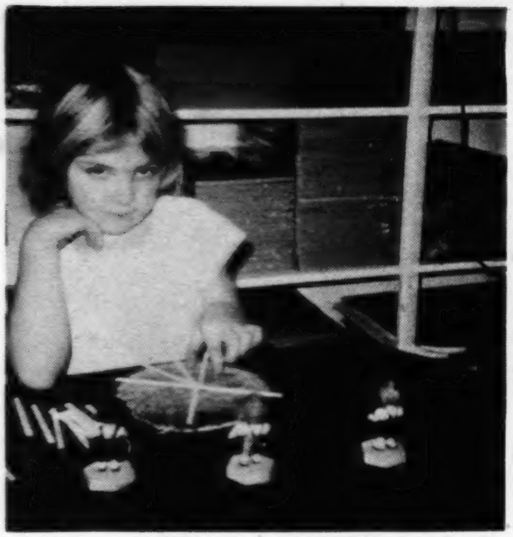


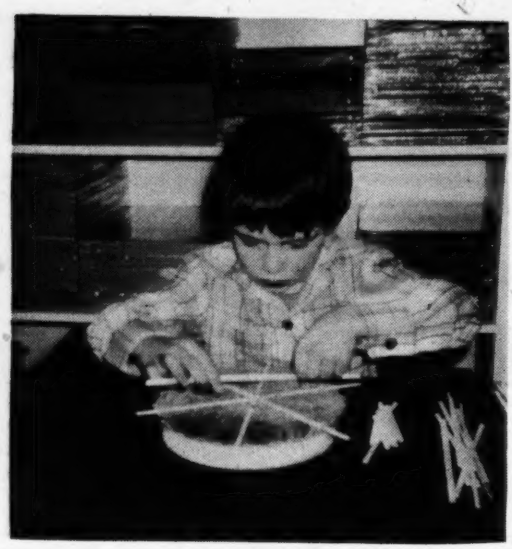
Figure 6

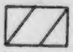
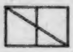
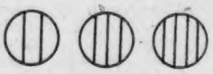
Stage III. The Evenness Stage

A third level of achievement is marked by a stage when the child attends to the size and shape of the parts and critically examines them to determine if they are equal or even. At this stage, partitions are correctly classified as "fair" or "not fair" although the child may not be able to produce equal shares (e.g., thirds or fifths).

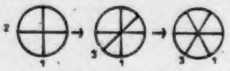
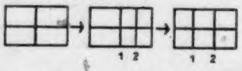
Prior to this stage, a child declares as "fair shares" partitions which are not equal. For example, if there are three people and the number of pieces produced is three, then it is declared a fair share. Another factor is the mechanical use of the halving algorithm. Because the algorithm "works" so well, it is used in a generalized manner, that is; it is used without careful scrutiny of the partitions it produces. Thus, a procedure which "works" in one situation is used in others even though it is inadequate or unsatisfactory. For example, on a rectangular region, a diagonal line can be used to partition evenly in two, so repeated "dichotomies" are performed in efforts to obtain fourths [▨ ▨]; the tool is used to obtain thirds


The Evenness Stage
"Fair shares" for 3 children.


The Evenness Stage
Obtaining Twelfths from Sixths

[]; or, two different halving lines are "mixed" []. Vertical parallel lines which "work" on a rectangular region are used on a circular region to produce thirds, fourths, or fifths []. In such instances, partitions are declared to be fair shares.

Stage three is a major breakthrough in the child's thought. The halving mechanism becomes more meaningful: equalness is now a critical characteristic in designating fair shares. Diligent attempts are made to construct congruent pieces and consequently partitions which do not "look the same" are designated to be "not fair".

Moreover, the equal concept, together with geometry transformations enable the child to produce a number of partitions other than powers of two. Even numbers of partitions can now be attained. For example, to obtain sixths, a child first partitions a circular region in fourths, then bisects two sections, and lastly, rotates two sticks to attain six equal parts []. On a rectangular region, sixths can be attained from fourths by translating a halving stick and placing an additional stick parallel to it []. In a similar manner, tenths are obtained from eighths; twelfths are obtained from sixths.

This primitive use of transformation geometry enhances the halving algorithm so that the child can now attain unit fractional parts whose denominators are even numbers. Although not tested, the halving mechanism is illustrative of two things: the doubling of a number of partitions and the attainment of additional parts by adding another stick and rotational moves.

In summary, the evenness stage is marked by two notions of evenness: a) the equality or evenness of parts, and b) the ability to partition unit fractions with even denominate numbers.

Stage IV: The Oddness Stage

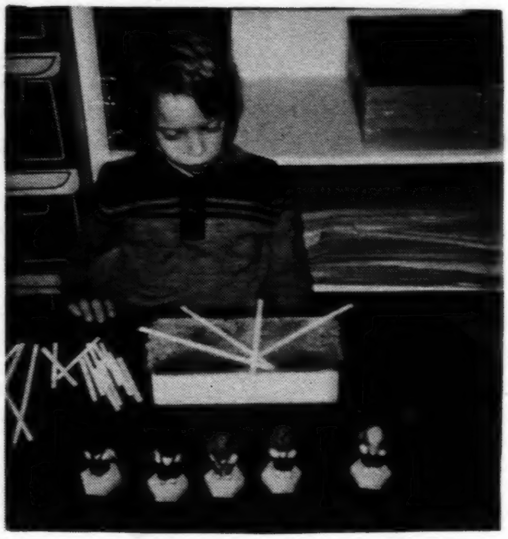
Following the evenness stage, there is a period when the child comes to be aware of the inadequacy of the halving mechanism. The cognizance of the limitations of the halving mechanism marks another significant breakthrough in the child's thought. At this stage, the child knows that by beginning with a halving line, some fractional parts cannot be attained efficiently on either a rectangular or circular region. The problematic numbers are three and five, or odd numbers. Some children label these as NOT EVEN or the HARD ONES. When asked to make an odd number of partitions, some children respond with statements such as, I CAN'T GET IT BECAUSE IT'S NOT EVEN or IT CAN'T BE DONE.

Doz (9;8) is an example of a child who has reached this stage. Working with a circular cake, she stated that it is "hard" to partition in three, five, and seven and readily offered an explanation: BECAUSE WHEN YOU TRY TO DO THREE, IT DOESN'T WORK. WHEN YOU TRY TO DO SEVEN, IT'S HARD BECAUSE IT MIGHT NOT WORK. "Not working" for Doz means IT WON'T FIT

The Oddness Stage
Circular region: 3 children
"Searching for a new first move."

The Oddness Stage
Rectangular region: 3 children
"Searching for a new first move."

The Oddness Stage
Rectangular region: 5 children
"Searching for a new first move."



The Oddness Stage
Rectangular region: 5 children
"Searching for a new first move."

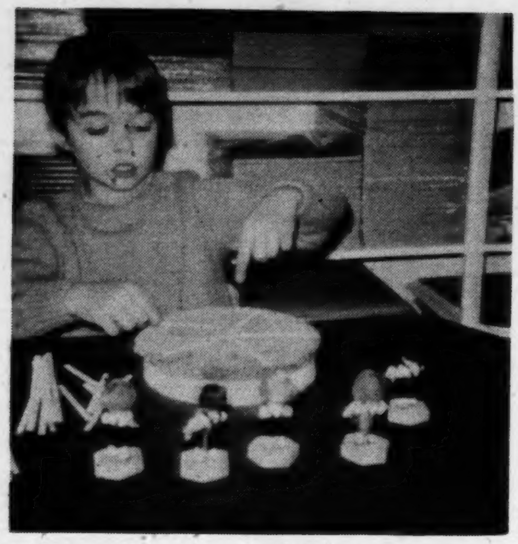
(see transcript, p.55). One of Doz's attempts to do thirds on the circular cake was with two vertical parallel lines. When asked if the sharing was fair she replied in the negative and explained: BECAUSE THIS IS THE CORNER AND THIS IS THE MIDDLE. This is evidence that Doz has passed through the evenness stage.

Bro (8;5) has also reached stage four. While partitioning a round cake, she stated that three and five were "hard to make". She was asked, "What's hard about three and five?" and she replied: WELL, IF YOU TRY TO MAKE SOMETHING OUT OF 'EM, YOU AREADY MADE THEM. She explained further: WELL, YOU TRY AND MAKE FIVE, BUT THEN YOU GET SIX. In attempting thirds she stated: LIKE IF YOU MAKE THREE LIKE THIS (a horizontal diameter cut and two vertical radius cuts) YOU'D GET FOUR, SO, LIKE IF YOU GO LIKE THIS (moving the sticks) YOU STILL GET FOUR. IF YOU GO LIKE THIS (one horizontal diameter cut), IT'S TWO HALVES (see transcripts, p.49).

Some of the children who had attained stage four were guided to discover the new first move. In such cases, the child's attention was brought to the need for a different first move with a comment like "Maybe it's how you begin" or "Could you begin differently?" Attempts with radius sticks were subsequently made. With some children further comments were made, for example: "You can get two and four. Now, look at the cake and see three." Despite this help, not all children succeeded at obtaining thirds and fifths. Bro and

Doz are two who did (see transcripts, pp. 49 and 55).

The Oddness Stage
Rectangular region: partitioning
in fifths



The Oddness Stage
Circular region: partitioning
in fifths

Realizing that a different initial cut is necessary to partition in thirds and fifths, the child actively searches for a new first move thus freeing his partitioning from the dominance of the halving mechanism. For a circular shape, the first move is a radius of the circle, whereas, for a rectangular shape, it is a vertical or horizontal line placed in a position other than the middle. The child uses this new tool in a counting algorithm. The algorithm frequently requires that a re-adjustment of the partitioning lines be made in order to obtain equal shares. Figure 7 depicts the algorithm as it is frequently used for obtaining fifths and sevenths.

Partitioning Behaviors: Fifths and Sevenths,

Circular and Rectangular Regions

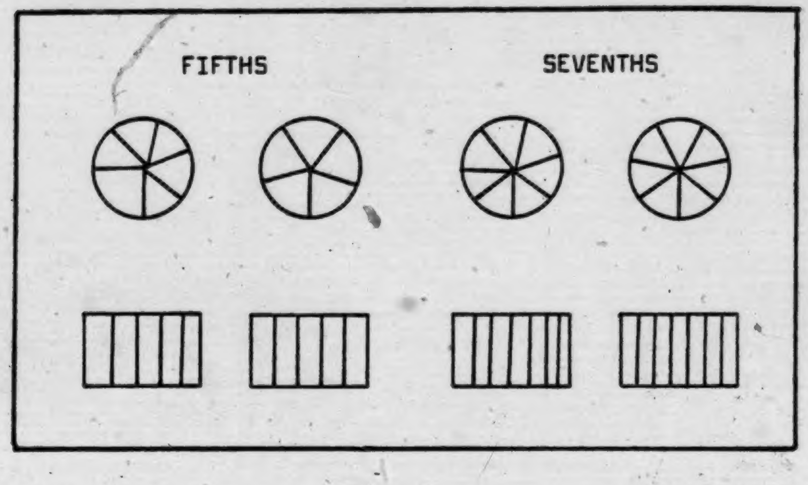


Figure 7

The counting algorithm enables the child to obtain fractional parts whose denominate numbers are odd; whence the name, oddness stage.

Only Jou (8;5) succeeded at obtaining thirds and fifths on both a rectangular and circular region on a first trial. Although Jou appeared to be confident of his methods, he did show some uncertainty during a second interview. At this time, he was demonstrably tired and he expressed a lack of sleep the previous night so he was not performing at his best.

From the sample, a total of six children were successful at partitioning both types of regions in thirds and fifths. With the exception of Jou (cited above), this was only achieved with guidance and after several trials on one or both regions.

Having attained the oddness stage, a child is able to partition rectangular and circular regions in any number of parts according to the child's counting ability and providing the region is sufficiently large to produce pieces of a reasonable size.

Stage V: The Primeness Stage

The oddness stage marked by the recognition of the inadequacy of the halving mechanism and the discovery of a new first move, enables the child to partition rectangular and circular shapes in any number of parts. As previously stated, the procedure used is a counting algorithm. This method is suitable for small numbers. However, it is an awkward algorithm for large numbers such as 9, 11, 13, 15, etc., because equal parts are not easily attained.

It is probable that with time, a child would question the one-by-one procedure and look for a different method. For numbers such as 11 and 13, there is, of course, no other method, but for 9 and 15, there is clearly a more efficient method.

To obtain ninths, a region can first be partitioned in thirds and then each part trisected. In like manner, fifteenths can be attained by trisecting fifths. The numbers 9 and 15 are composite. What is being done is that the region is first partitioned in a number of parts that is equal to one of the number's prime factors and then each part is subdivided by the number that is the other prime factor. Composite numbers whose prime factorization is three numbers would necessitate a three-step partitioning process. For example, to obtain eighteenths, the procedure could be: first, partition the region in half; next, each half in thirds; and lastly, each sixth in thirds. Other sequences could be tried; for example, first, by partitioning the shape in thirds, then bisecting or trisecting each third, and lastly, partitioning each part attained by the remaining prime factor.

The alternative sequences, depicted in Figure 8, demonstrate that the choice of prime factor for any move does not affect the results. This is an illustration of the Fundamental Theorem of Arithmetic.

A child who proceeds to partition composite numbers in this manner employs a multiplicative algorithm. Such a child has reached the primeness stage.

The primeness stage is the last stage in the development of partitioning as a constructive mechanism in rational number learning. All the unit fractions can now be attained; the foundation is complete for partitioning a continuous whole. Other partitions, that is, non-unit fractions, are merely combinations or multiples of the unit fractions.

The unit fractions with prime denominators as the foundational or primitive elements are of primary importance in attaining mastery of the partitioning mechanism; therefore, they are assigned the label of primal fractions.

Partitioning in Eighteenths: Circular Region

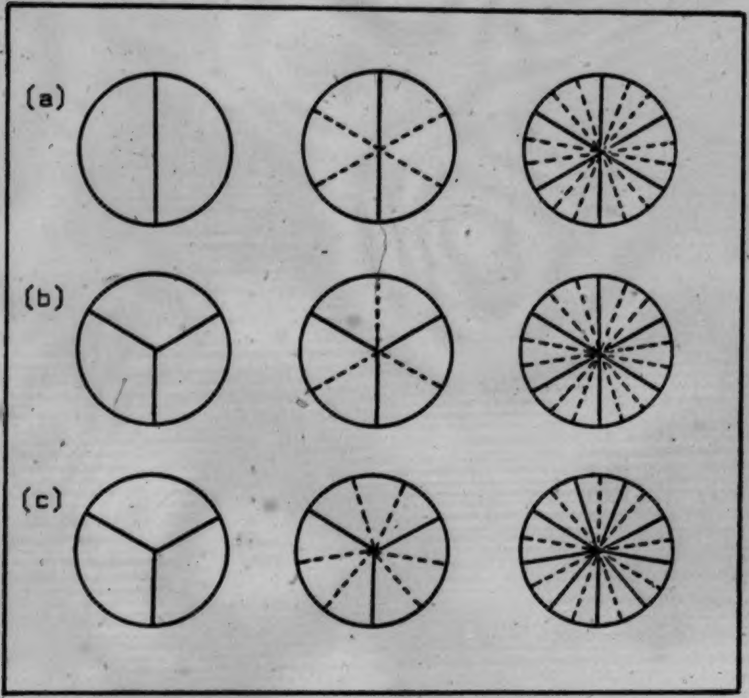


Figure 8

Conclusion

It has been demonstrated that the partitioning mechanism is mastered through the successive attainment of four subsets of the unit fractions. These are: the primal fractions (denominators are prime numbers), fractions whose denominators are powers of two, even fractions, and odd fractions.

The primal fraction one-half in the first learned mechanism. Mastery of this mechanism enables the child to attain a subset of the unit fractions, namely, fractions whose denominators are powers of two. Later, with the aid of geometry transformations the child is able to partition a larger subset of unit fractions; that is, the child can attain all fractions whose denominators are even.

The primal fraction one-third marks a major advancement in the child's partitioning capability. With this achievement, a child can partition unit fractions whose denominators are odd.

Lastly, use of prime factors enables the child to efficiently partition unit fractions whose denominators are composite numbers.

V. SUMMARY, DISCUSSION, AND RECOMMENDATIONS

A. Summary of the Investigation

The partitioning behaviors of young children were the focus of the present study; a focus oriented towards the discovery of insights regarding the nature of the behaviors manifested and their role in the development of the partitioning mechanism.

The study, by design, involved the researcher interacting with children in the context of controlled yet flexible clinical sessions. By means of intense observation, questioning, dynamic involvement, and ongoing sampling and analysis, the possibility of gaining insights into the meaning of the partitioning behaviors of children was maximized.

Five partitioning tasks were devised for the study, each one characterized by materials differing in substantive nature, varying in mathematical difficulty, and admitting modifiable solution procedures.

In this chapter, the sample, the tasks, and the major conclusions are summarized. This is followed by a discussion of the research methodology and the findings. Lastly, recommendations for further research are suggested.

The Sample

The final sample consisted of 43 children chosen in emergent fashion by the researcher from two kindergartens (half day classes), a grade one, two, and three class in a public primary school within the Yellowhead School Division, Alberta. Four participating teachers were selected by the school principal after the purpose of the study and the research methodology had been presented and explained.

Following the method of theoretical sampling (Glaser & Strauss, 1967) and in accord with the purpose of the study, the sampling process extended throughout the period of data collection. The selection of children was not guided by intelligence or performance scores but, by the researcher's assessment that the particular child would freely interact during a clinical session. This decision was made while observing and interacting with the children in their classrooms prior to the time of data collection.

The total sample was composed of 22 girls and 21 boys and included 8 kindergarten, 8 grade one, 12 grade two, and 15 grade three children. The mean chronological age of the children was 7.12 years.

The Tasks

The five partitioning tasks which were devised and used exemplified distinct characteristics for the purpose of the study. These particular qualities are:

a. The tasks embodied materials which varied in substantive nature, namely: discrete; discrete set with elements divisible; discrete set with subsets separable; continuous; and, continuous but divisible.

b. The mathematical aspect of the task varied, enabling all the children to experience success while providing the opportunity to challenge their thinking.

c. Each task allowed for alternative and easy modification of a solution.

d. The task settings and materials were believed to be familiar and motivational to the children. The manipulation of materials as a method of solution was thought to be a mode conducive for the children to demonstrate their capabilities.

Conclusions

The present study was framed within a discovery paradigm; its focus was the discovery of insights regarding the development of the partitioning mechanism through an understanding of the behaviors young children exhibit while solving selected partitioning problems.

Since the major findings supercede the specificity of the behaviors manifested while interacting with each problem, the results have been reported and are summarized here from a holistic view relative to the development of the partitioning mechanism.

From insights gained and a systematic analysis of the data, a theory concerning the development of the partitioning mechanism has been formulated and presented. The key cognitive movements and characteristic concepts marking the progressive development of the partitioning mechanism are presented in the form of propositions; propositions which comprise the substantive elements of the theory.

Proposition I

The conceptual structures necessary to achieve mastery of the partitioning mechanism can, in theory, be effectively expressed using elemental ideas of number theory and transformation geometry. Number theory concepts which play a key role are: even and odd, prime and composite, factors and multiples; geometry notions are translations, rotations, symmetry, similarity, congruence, and the structural dimensions of the regions.

At times, the child appears to engage in a mental "struggle" with one or more of these ideas as he or she reaches out to construct different partitions on rectangular and circular regions. At other times, there appears to be a kind of "selective focus", whether consciously or not, as the child seemingly concentrates on one aspect while overlooking others.

Proposition II

Further to the new knowledge concerning mental constructs which undergird the partitioning mechanism, a hierarchy of behaviors in the development of the mechanism has been identified.

Within the theory, five stages of partitioning capabilities are proposed: Four stages are grounded in the data, while a fifth stage is proposed as a logical next level and necessary for mastery of the partitioning mechanism.

The proposed stages are re-presented here in point form while highlighting three distinctive characteristics, namely:

a. the operator or key concept developing during the stage;
b. the algorithm or procedure employed to produce the partitions; and
c. the domain or the extent of the partitioning capabilities within the stage.

The stages are:

Stage I: The Sharing Stage

Operator: - breaking; sharing; half it.

Algorithm: - allocating pieces ("a piece for you").

Domain: - social setting; counting numbers [$\{C\}$].

Stage II: The Halving Algorithm Stage

Operator: - partitioning in two; halving algorithm.

- no notion of equality.

Algorithm: - repeated dichotomies.

Domain: - one-half and other unit fractions whose denominate numbers are powers of two [$\left\{ \frac{1}{2^n} \mid n \in C \right\}$].

Stage III: The Evenness Stage

Operator: - equalness; congruency.

- halving algorithm becoming meaningful.

Algorithm: - halving algorithm; geometry transformations.

- extension of the halving algorithm to doubling any partition and adding two parts.

Domain: - unit fractions with even denominate numbers [$\left\{ \frac{1}{2n} \mid n \in C \right\}$].

Stage IV: The Oddness Stage

Operator: - evenness; oddness.

- search for a new first move.
- use of the new first move
- geometry transformations.

Algorithm: - exploratory measures; trial and error.

- counting; one-by-one procedure.

Domain: - all unit fractions [$\left\{ \frac{1}{n} \mid n \in C \right\}$].

Stage V: The Primeness Stage

Operator: - composition of numbers (Fundamental Theorem of Arithmetic).

- use of prime factors.

Algorithm: - multiplicative.

Domain: - all unit fractions

$[\{ \frac{1}{n} \mid n \in C \}]$.

B. Discussion

The Research Methodology

The study was a concentrated effort to discover foundational concepts in the development of the partitioning mechanism relative to rational number development. This quest for understanding the nature of the partitioning behaviors of young children necessitated a clinical research technique within a discovery paradigm.

The discovery paradigm demands that one "interact with the phenomenon" so that the resulting theory will "express the essence of the phenomenon", making it meaningful (Sawada,1980). Accordingly, a clinical interaction technique modeled on Piaget's clinical interview was designed for the study. The process involved the researcher in intensive interactive sessions which saw young children engaging in activities which require constructive processes deemed important in the child's acquisition of rational number (Kieren, 1980 a; Piaget et al., 1960).

The mode of investigation is affirmed as a powerful tool in uncovering conceptual structures which undergird concepts, in this case, the partitioning mechanism. The value or validity of the methodology is affected positively or negatively by the inter-acting participants (researcher and respondent), the event (the task situation), and the analysis process. These three agents merit a brief discussion here.

The clinical researcher is distinguished by certain personological qualities and capabilities (Good, 1959; Piaget, 1929; Posner & Gertzog, 1979; Wilson, 1977). Pre-training and experience in observing; that is, in looking, listening, and responding to young children are deemed mandatory. In addition, a degree of faith in the ability of a researcher to carry out fruitfully this type of investigation appears to be necessary.

The respondent's behavior is affected by the researcher, the environment (of the clinical session) and the relevancy of the situation to which he or she is to react. In this study, the atmosphere was warm and supportive and the children appeared to get meaning out of what they were doing.

The problem setting (birthday party) was familiar to the children; the materials were attractive and captured their attention; the task of sharing a cake, cookies, and candies made sense to them. Moreover, the children experienced success; they could all partition the objects in one way or another. Consequently, the challenge of "doing it [partitioning] a different way" was generally accepted with joy and eagerness.

The strength of the discovery paradigm resides also in the analysis process which "should blur and intertwine continually from the beginning of the investigation to its end" (Glaser & Strauss, 1967, p.43) with the data collection and coding operation.

In this study, data analysis continued through three stages; during clinical interactions, between clinical sessions (daytime, evening, and weekend), and following the termination of the data collection. The first and second stages are of critical importance and determine the extent to which the paradigm will "work". For this reason, the researcher must try to analyze all information from the initial moment with sincerity and assiduity.

A final word which must be said about the paradigm relates to time. The analysis process is demanding. Only a few clinical sessions can be conducted per day; otherwise, fatigue may reduce the researcher's effectiveness. Thus, a researcher engaged in this method of investigation must be prepared to devote a considerable length of time before bringing a study to a satisfying conclusion.

The Findings

The findings are discussed under the following headings:

1. Development of the partitioning mechanism: the proposed stages.

2. The effect of the selected geometric regions.
3. The dominance of the halving mechanism.
4. Order of achievement of unit fraction partitioning capability.

Development of Partitioning Mechanism: the Proposed Stages

It is proposed that five stages must be attained before mastery of the partitioning mechanism is achieved. In this study, the age range of the children was from 4 years, 11 months to 9 years, 8 months. None of the respondents had attained the last stage; only one had reached stage IV, that is, he could partition a rectangular and circular region in any number of equal sized parts.

Looking at the grade level of the children who were successful at partitioning the various unit fractions, it is difficult to assign a stage-age level. The value of doing this is questioned. What appears to be important for the children to achieve success is their previous experiences or learning opportunities. Admittedly, maturation is thought to play a role in concept attainment. Some respondents of different age levels were "ready" to move on to a next stage. This is evidenced by the fact that some "discovered" for themselves how to partition in odd numbers while others were prompted to do so. What appeared to be needed was an experience which "called them forth" to demonstrate their capability, or, as Wheeler (1970) expresses it, "to throw [them] to the edge of [their] resources" (p.27).

Difficulty Level of the Selected Geometric Regions

Rectangular and circular regions were chosen for the partitioning instrument because of familiarity but also to assess the difficulty level of each shape as a region to be divided.

Although it has been claimed that young children find it easier to recognize fractional parts of circular wholes than of other regions (Gunderson & Gunderson, 1957), the findings indicate that young children are more successful at partitioning rectangular shapes than circular ones.

The percentage of successful partitions was consistently as high or higher for the rectangular region than for the circular region.

Partitioning in halves and fourths on both regions was within the capabilities of a large majority of the children. The results are different for partitioning in thirds and fifths. While 64 percent of the children were successful at partitioning a rectangular region in thirds, only 32 percent were successful at thirds on a circular region. Partitioning in fifths was successfully attained by 62 percent of the children on the rectangular region and by 33 percent on a circular region. Such discrepancies appear to indicate that a circular region is more challenging than a rectangular region in attaining odd numbers of partitions.

Dominance of the Halving Mechanism

In attempts at partitioning both rectangular and circular regions in thirds and fifths, the dominance of the halving mechanism was evidenced. Many children seemed incapable of deviating from employing a halving line as the initial "cut". A few who began with a line other than a median one only did so after several trials or with guidance.

The literature (Hiebert & Tonnessen, 1978; Piaget et al., 1960) reveals that young children master halves and fourths prior to thirds; therefore, the children were first asked to partition the regions in an even number of equal sized parts, usually in two pieces. The question arises if the results would have been different if partitioning in an odd number of parts had been the first activity. Would the halving mechanism have been so dominant or is this behavior a sequence effect? Further research could provide an answer to this query.

Order of Achievement of Unit Fraction Partitioning Capability

Five stages in the development of the partitioning mechanism were identified. These demonstrate that a child is able to partition first in half; then, with the acquisition and eventual mastery of the halving algorithm, in powers of two; thirdly, with the use of geometry transformations, in even numbers.

Partitioning in odd numbers follows the learning of a first move other than a median cut. With the discovery of the "new first move" children are able to partition in thirds, fifths, and other odd numbers; thus, thirds and fifths are achieved in concert. The algorithm used is a counting one and equality of parts is usually achieved by rotational and translation moves.

Some children attain fifths before thirds. There appears to be a reluctance at constructing pieces smaller than fourths.

For mastery of the odd numbered partitions, number theory ideas of odd and even appear to be prerequisite. A child who attains thirds or fifths and not both is not cognizant of the oddness quality; therefore, the success is only accidental. Subsequent attempts at either thirds or fifths may prove unsuccessful.

The Significance of the Stages

In discussing mathematics education research and its influence on educational practice, Carpenter (1980) states that a major objective should be "to characterize the processes and concepts that children acquire at significant points in the learning of important mathematical topics" (p.194). Furthermore, Carpenter asserts that the results of such research can influence educational practice in two ways: it can affect the selection and sequencing of content and it can contribute towards individualizing instruction on the basis of a child's development level concerning certain concepts and processes (Carpenter, 1980).

The goal of the present study was to uncover underlying mental structures of the partitioning mechanism as it relates to rational number development. From an interpretation of the partitioning behaviors of young children, a theory has been proposed wherein foundational concepts are identified and stages characterizing distinct levels of partitioning capabilities are delineated. Within each stage, developing concepts and processes, together with partitioning capabilities, are particularized. This knowledge is perceived to be highly significant in considering pedagogical issues relative to rational number teaching and learning.

The findings can provide valuable guidance in curricular and instructional decisions regarding the kinds of experiences to provide for young children; experiences which will serve to build a firm foundation for later rational number learning.

C. Recommendations for Further Research

The findings of this study generate a number of questions for further research. These are presented below.

1. Observation: The present investigation employed rectangular and circular regions as areas to be partitioned.

   Research Question: What would be the behaviors of young children while partitioning triangular, pentagonal, or n-gonal shapes?

2. Observation: The findings revealed that the structural variances of the two regions challenged some of the respondents in their partitioning attempts.

   Research Question: What would be the partitioning behaviors if children were presented with only one type of region?

3. Observation: In the study, children were first asked to partition the regions in an even number of equal sized parts, usually in half. The halving mechanism was found to dominate their thinking and so affect their partitioning behaviors.

   Research Question: Would the dominance of the halving mechanism persist if a sequence of odd numbers rather than even was employed first?

4. Observation: Elemental notions of geometry were noted to play a key role in the child's achievement of mastery of the partitioning mechanism.

   Research Question: How do geometry ideas of similarity, congruence, and symmetry affect the partitioning behaviors of children?

5. Observation: Basic number theory ideas of even and odd, prime and composite were noted to play a significant role in the behaviors of children as they attempted to partition regions in 2, 3, 4, 5, or more equal sized parts.

Research Question: Would prior experiences and explorations with basic number theory ideas enhance mastery of the partitioning mechanism?

6. Observation: The setting for the partitioning activities was a birthday party and the children simulated partitions of familiar objects.

   Research Question: Would the partitioning behaviors be similar in other or less familiar situations?

7. Observation: Stages in the development of the partitioning mechanism are identified as a result of insights gained from clinical interactions with children of age range 4;11 to 9;8.

   Research Question: Would a longitudinal study of children over a same time span (4 to 5 years) reveal corresponding findings?

D. Concluding Statement

The goal of the present study was understanding; its achievement resides in the proposed theory regarding the development of the partitioning process. The concepts and propositions of the theory emerged from the total experience of a series of clinical interactions with young children.

The intent has been to develop a meaningful theory, the components of which will be pondered, questioned, analysed, and tested by various means. The extent to which the theory has captured the essence of the partitioning process in young children will determine the value of the study.

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APPENDICES

APPENDIX A

LETTER TO PARENTS

University of Alberta
Edmonton, Alberta T6G 2G5
October 28, 1980

Dear Parent:

During the next several weeks, I will be interviewing children at A. H. Dakin School as part of a research project in mathematics education. I have chosen to videotape the interview sessions as the best method of recording the data for subsequent analysis. Whenever the tapes will be viewed, the identity of the children will always be withheld.

The signatures of the superintendent of the Yellowhead School Division and the principal of A. H. Dakin School, affixed to this letter, will attest to the fact that I have their permission to conduct this research study.

If you have no objection to your child taking part in the study, would you kindly fill in the attached form and have it returned to your child's teacher.

Sincerely yours,
S. Yvonne Pothier
Sister Yvonne Pothier

Klaus Puhlmann
Superintendent,
Yellowhead School Division

Principal,
A. H. Dakin School

Please detach and return to your child's teacher.

I am willing to have my child, ____________________ , take part in the research project in mathematics education should he/she be selected to be interviewed.

____
Date

____
Parent's signature

APPENDIX B

DATA SHEET

DATA SHEET

DATE: November 26, 1980 NAME: # 20 GRADE: 2
TIME: 9:10 - 10:10 BIRTHDATE: 08-07-73 AGE: 7;4
Break: 9:45

CARTON-TRUCK PROBLEM

15 cartons
1] 8 on T1, 7 on T2
2] 2 on each, +2, +1 same pattern
3] 5 on each at a time " "
4] " " " " " " " "

17 cartons
1] 6, 6, 5 -> 5, 5, 5

COOKIE PROBLEM

4 children, 16 cookies
1] 1 by 1 same pattern
2] 1 by 1 " "
3] 4 at a time stacked them

4 children, 10 cookies
1 by 1 "Split 'em in half" "2 and a half"

4 children, 9 cookies
1 by 1 Split 'em in 4. Broke off 1 piece, then another – irregular
"2 and a half" (definition of half)

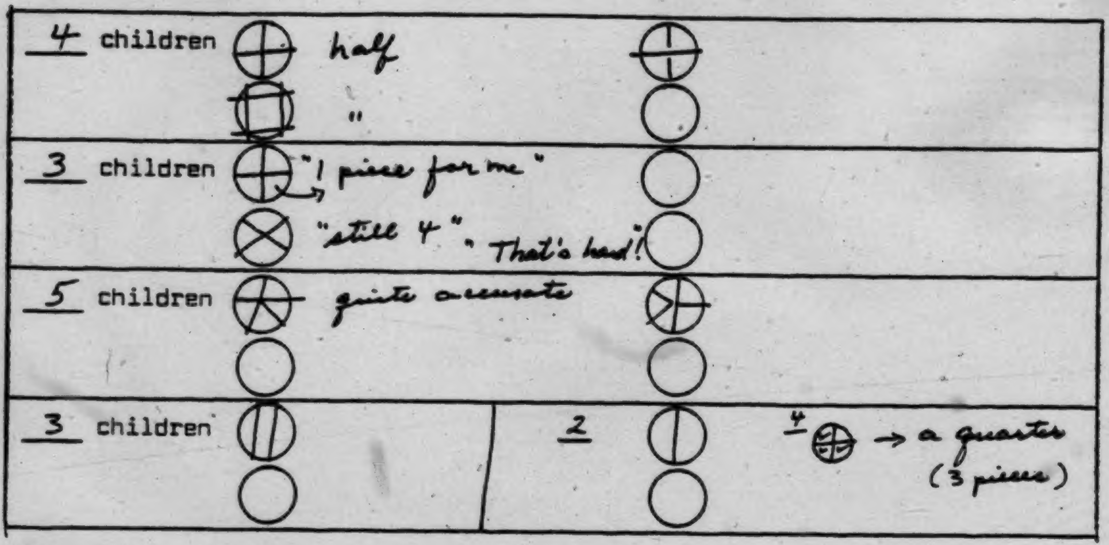
3 children, 10 cookies
1 by 1 Split it in 3. (irregular)

3 children, 8 cookies
I'm halfing that & ...
Broke each in 3. "2 whole O a 2 half –.

DATA SHEET

NAME: #20 GRADE: 2 AGE: 7;4

GIANT COOKIE



4 children — half

"

3 children — "1 piece for me"

"still 4". That's hard!

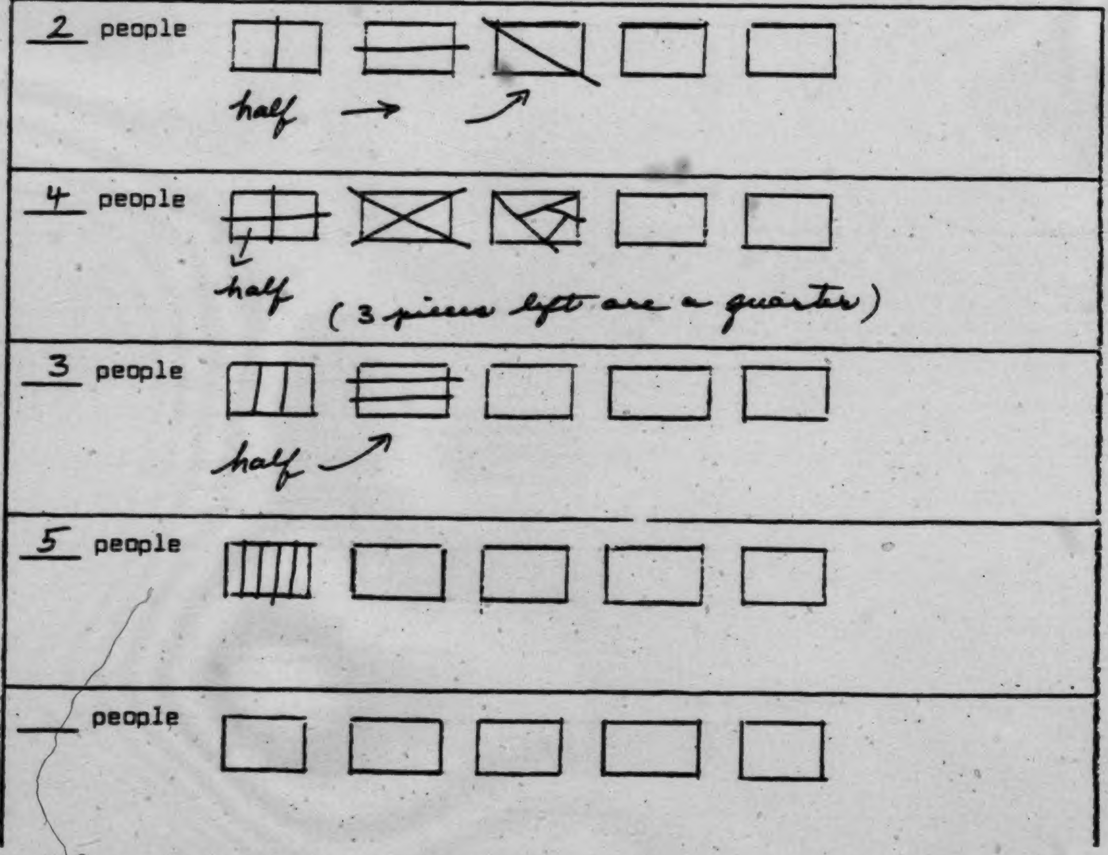
5 children — quite accurate

3 children

2

4 → a quarter (3 pieces)

CAKE PROBLEM



2 people

half → ↗

4 people

half (3 pieces left are a quarter)

3 people

half ↗

5 people

___ people

DATA SHEET

NAME: # 20 GRADE: 2 AGE: 7;4

BOXED-CANDY PROBLEM

2 children, 7 boxes ✓

3 + a half

4 children, 9 boxes

2 + a half

3 children, 7 boxes

2 + a half

CHOCOLATE BAR PROBLEM

2 children, 12 pieces

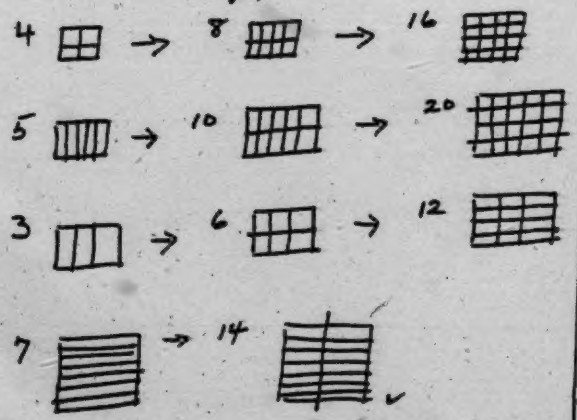
half

3 children, 12 pieces ✓

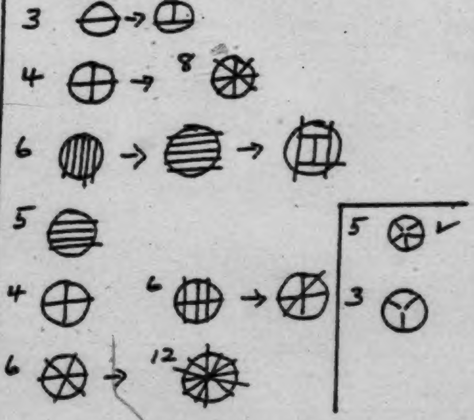
4,4,4 ⊞

___ children, ___ pieces

Cake (large)


4
8
16
5
10
20
3
6
12
7
14

Round Cake


3
4
8
6
5
4
6
6
12
5
3

APPENDIX C

SELECTED TRANSCRIPTS

The Carton-Truck Problem

Wal (5;0)

3 trucks; 15 cartons

"Can you load the cartons onto the trucks so that each one will have as much?" Wal placed four cartons on each truck, then added one more. "Do they each have as much?" She counted the cartons out loud. YES. "Show me another way to load the cartons so that they'll each have as much." I DON"T KNOW ANOTHER WAY. "Would you like to try it?" She again placed four cartons on each truck, then added one more. "Do they have as much now?" She counted them aloud again. YES. There was no evident pattern in the way she placed the cartons on the trucks.

3 trucks; 17 cartons

Wal placed the cartons on the trucks one by one and counted aloud as she placed them: 1, 1, 1; 2, 2, 2; 3, 3, 3; 4, 4, 4; 5, 5. She stopped counting, but placed the remaining cartons on the trucks ending with groups of 6, 6, and 5. "Do they each have as much?" NO. "Tell me what's happening." THERE'S ONLY TWO MORE LEFT. AND THIS ONE DOESN'T HAVE THE SAME. "That's right. So, what could you do to make them all the same?" NEED ONE MORE. "Suppose we didn't have any more. What do you have to do to make them all the same? TAKE TWO AWAY. "Why don't you do that." She did. "And now, do they have the same?"

YES.

The Cookie Problem

Gib (6;5)

4 children; 16 cookies

"Would you give the cookies to the children so that they each have as much?" Gib placed five cookies on each of the first three plates. I JUST HAVE ONE MORE LEFT. "So what can you do? We want them to have as much." He changed the cookies to four on each plate. "Alright, what's happening now?" THEY'RE ALL THE SAME. "How much cookie?" FOUR.

"Share the cookies with the children another way." He placed four cookies on each plate; the same pattern on each plate. "Do they each have as much?" YES. "Let's remove them. Share them a different way." He placed four cookies on each plate but stacked them. FOUR. "What's happening this time?" I PUT THEM IN A PILE. "Do they each have as much?" YES. "Share them another way." He placed four on each plate, but arranged them differently.

4 children; 14 cookies

Gib placed three cookies on each plate, stacking them. THREE! "Is it a fair share?" YAP. "Have you used all the cookies?" NO. "What's happening?" IF I GAVE TWO MORE TO TWO MORE KIDS, THEY WOULDN'T HAVE THE SAME.

"What could be done to share those cookies with the children?" SPLIT 'EM. "Would you do that?" He broke one cookie in eight pieces, the other in four pieces. SIX. "Pardon me?" SIX. "Six what?" COOKIES. "Where?" ONE, TWO, THREE, FOUR, FIVE, SIX. He counted to six for each plate. "So, if this little girl was to go home and tell her mother how much cookie she had eaten, what could she tell her?" SIX. "Six what?" SIX COOKIES. "Would she have eaten six cookies?" EMHEM. "Six whole cookies?" NO. "What would she have eaten?" THREE WHOLE COOKIES AND THREE HALF COOKIES. "What about this little boy?" THREE WHOLE COOKIES AND THREE HALF COOKIES. He stated the same amount for each child. "Why do you call these 'half cookies'?" CAUSE THEY'RE SPLIT IN HALF.

Suppose you had just these 2 cookies to share with the children. How would you do that? SPLIT 'EM. "Do that." He again broke one cookie in 8 pieces and the other in 4 irregular pieces. THREE. "How many?" THREE. "How much cookie would this girl have eaten?" THREE. "Three what?" HALF. "If she had eaten just this much (one piece), how much cookie would she have eaten?" ONE. "One what?" HALF. "And if she had eaten these pieces (2 pieces)? TWO. "Two what?" HALF. "And now, if she had eaten this one, this one, and that one?" THREE HALFS.

4 children; 9 cookies

Gib gave two cookies to each child. "Tell me what you're going to do there?" SPLIT. "How?" IN FOUR LITTLE PIECES. "Okay, go ahead. I just wanted you to tell me." [illegible] broke the cookie in four irregular pieces. THREE. "Is that a fair share?" YES. "Does this girl have just as much cookie as this boy?" EMHEM. "How much cookie does this little girl have?" THREE. "Three what?" THREE COOKIES. "Does she have three cookies?" EMHEM. "Does she have three whole cookies?" NO. "What does she have?" TWO WHOLE COOKIES AND ONE HALF. "Alright, and this boy?" TWO WHOLE AND ONE HALF. "So, how much cookie does this boy have?" TWO WHOLE AND ONE HALF.

3 children; 10 cookies

Gib gave two cookies to each child. GONNA SPLIT THIS ONE. "How?" IN THREE HALVES. "Okay." FOUR. "Four what?" COOKIES. "How much cookie does this little boy have?" THREE WHOLE ONES AND ONE HALF. "Okay, let's remove that."

2 children; 5 cookies

He placed two cookies on each plate. SPLIT THIS ONE. "How are you doing that?" IN TWO PIECES. "How much cookie does that girl have?" THREE AND ONE HALF. "Pardon?" TWO WHOLE ONES AND ONE HALF. "And that one?" TWO WHOLE ONES AND ONE HALF. "Why do you call this 'one half'?" CAUSE IT'S SPLIT IN HALF. "What does that mean?" I DON'T KNOW.

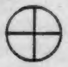
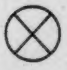
The Cake Problem: Circular Region

Dei (9;2)

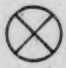
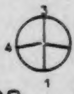
4 children

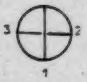
In this instance, the Giant Cookie was used instead of the round cake.

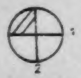
"Share the cookie with these children so that it's a fair share."

 "Would that be a fair share?" YES. "How much cookie would he get?" A QUARTER. "Show me another way."  "Fair share?" EMHEM. "How much cookie does this boy get?" A QUARTER. "Show me another way." He made no other attempt.

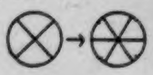
3 children

"Do it so that it's an even share for these children."  NO. 

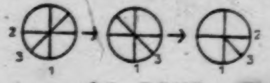
YOU CAN'T DO IT. BUT STILL. NOW I KNOW WHAT TO DO. 

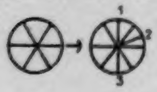
YOU CANNOT DO IT BUT LEAVE ONE OUT. "Fair share?" EMHEM. "How much cookie does he get?" THIS MUCH. (He pointed to one piece.) "Is all the cookie used up?" NO. "Try to do it so that all the cookie is used up and it's a fair share." 

"Fair share?" YA. I WOULD SAY. YA. "How much cookie does this boy get?" ONE HALF AND A QUARTER. A LITTLE BIT MORE THAN HALF. "Show me her share." THIS ONE AND THIS ONE. (HE pointed to a quarter piece and a small piece.) "How much cookie is he getting?" HALF AND A QUARTER. ALL OF THEM ARE THE SAME. "What part of the cookie is this (a quarter piece)?" HALF. NO, A QUARTER. "And this (a small piece)?" A LITTLE BIT LESS THAN A QUARTER.

"Try it again to see if you can use all the cookie and they each get one piece."  He closed his eyes and made gestures with his hands as if trying to visualize the parts. I CAN'T DO IT.

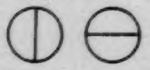
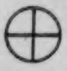
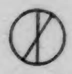
5 children

"Share the giant cookie with these children so that they each have as much."


I CAN'T DO IT. "How many pieces do you have?" FIVE. TWO OF THEM WOULD NOT GET THE SHARE. "Try it once more." 

I HAVE FOUR EXTRAS.

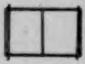
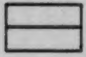
2 children

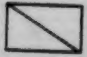
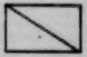
 "Fair share?" YES. "How much?" HALF. "What about more than one piece."  "How much cookie does she get this time?" A HALF STILL. "What's another way to call her part?" TWO QUARTERS. "Show me another way."  THIS WAY. THE BOY WOULD GET A LITTLE OF A HALF AND THE GIRL WOULD GET THIS AND ANOTHER HALF.

The Cake Problem: Rectangular Region

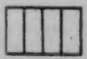
Pau (9;0)

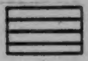
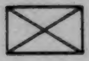
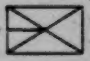
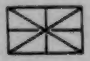
2 children

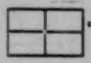
"Share the cake with the children so that they each have as much."  "How have you shared the cake?" I GAVE A QUARTER TO THIS ONE AND A QUARTER TO THIS ONE. "Show me another way to share that cake with the children."  A QUARTER TO THIS ONE AND A QUARTER TO THIS ONE. "How have you cut the cake?" LIKE STRAIGHT ACROSS. "How much cake does he get?" A QUARTER. "Why do you call it that?" CAUSE IT LOOKS LIKE A WHOLE CAKE, BUT IT'S A QUARTER.

"Show me another way."  "Is that a fair share?" YA. "And how much cake does he get?" A QUARTER. "Show me another way."  "Fair share?" YES. "How much cake does he get?" A QUARTER. "Show me another way." I CAN'T GO ANOTHER WAY. "Alright, we'll change the cake and we'll pretend that these children arrive at the party."

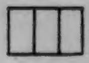
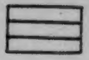
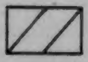
4 children

"Share this cake with these children."  "Fair share?" YAP. "Where is his share?" "How much cake does he get?" A HALF, QUARTER I SHOULD SAY. "Why do you call it that?" CAUSE IT LOOKS LIKE IT'S A HALF A CAKE. BUT IT AIN'T. IT'S A QUARTER.

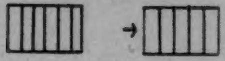
"Show me another way."  "Fair share?" YES. "And how much cake does he get?" QUARTER. "Another way."  "Fair share?" YES. "How much?" QUARTER. "Why do you call it that?" CAUSE IT LOOKS LIKE IT'S HALF A CAKE. "What does that mean?" IT MEANS THAT--IF I PUT ONE THERE [→ ]IT'D BE TWO SO THAT'S A WHOLE CAKE. "Where is the whole cake?" RIGHT HERE (pointing to the cake). I GIVE 'EM HALF RIGHT THERE AND HALF RIGHT THERE AND HALF RIGHT THERE []. I CUT IT IN HALF LIKE THAT. "So, how much cake does he get?" A HALF. "And if the girl gets this part (one eighth), how much cake is that?" HALF. He called each part HALF.

"Share the cake another way."  "How much cake is this (one piece)?" HALF. He called each part a HALF. "How many 'halves' there?" FOUR.

3 children

"Share this cake with these children so that they each have as much."  He did this without any hesitation. "Fair share?" YES. "How much cake does she get?" HALF. "Show me another way."  "How much do they each get?" HALF. "How many halves in the cake?" THREE. "Another way."  "Fair share?" YAP. "How much cake do they get?" HALF. "Alright. Let's share the cake with all these children."

5 children

 "Is that a fair share?" YA. "How much cake does she get?" HALF. "How many halves in that cake?" FIVE.

Further probing revealed that when a median stick was placed on the cake, Pau described this action as CUTTING IT IN HALF. However, when asked how much cake each share was, he called it A QUARTER. With additional examples, Pau used the term half correctly in some instances when describing partitions.

The Chocolate Bar Problem

Sche (8;6)

2 children; a 12 piece (2x6) chocolate bar

"Share the chocolate bar with these children so that they each have as much." She gave two pieces to each child at a time. "How much bar do they each get?" SIX. "Could you say that another way?" HALF. "Share the chocolate bar with them another way." She gave them one piece at a time. "How much chocolate bar do they get?" HALF.

3 children; a 12 piece chocolate bar

Sche gave the children one piece at a time. "How much chocolate bar does this girl get?" SHE GETS A QUARTER. She called each share a QUARTER.

4 children; a 12 piece chocolate bar

Sche shared the chocolate bar one piece at a time. "How much bar does the girl get this time?" HALF. QUARTER. "Pardon?" THIRD. She called each share a THIRD. "Would you tell me why you're calling that a third?" WE LEARNED IT IN CLASS. IN THREE, IT'S A THIRD.

If I were to share the chocolate bar this way (a vertical median cut), what am I doing?" YOU ARE SHARING A HALF. "This way (a horizontal cut), what am I doing?"

SHARING IT IN HALF. "And if I were to share it this way (in fourths) with that many (four) people, what have I done?" YOU PUT THREE FOR EACH PEOPLE. "Yes, and how much chocolate bar is that?" A THIRD.

The chocolate bar was then divided in three equal parts. "How much chocolate bar is this share?" QUARTER. "Why would you call that a quarter?" BECAUSE THEY HAVE FOUR AND A QUARTER IS FOR FOUR.

The Boxed Candy Problem

Toe (9;1)

2 children; 7 boxes of candy

Toe gave three boxes to each child and held on to the seventh one. "What can you do with that one?" I CAN'T BREAK IT IN HALF. "Why not?" BECAUSE IT'S NOT A COOKIE. "You could open the box and share them." YA. "Show me what would be his share." She pointed to half of the box. "How many boxes of candy would he have?" THREE AND A HALF. "And how much would she have?" THREE AND A HALF.

4 children; 9 boxes of candy

"What's going to happen this time?" I'M GOING TO GIVE EACH ONE A STRIP. "What would be their share of that box?" THIS STRIP WOULD BE HIS, THIS STRIP WOULD BE HERS, AND THIS STRIP WOULD BE HERS. "How many boxes of candy would he have?" TWO AND A HALF. She said this for each share. "So how many halves are there in that box?" FOUR.

3 children; 7 boxes of candy

"How many boxes of candy do they get?" TWO. "And this one (the seventh box)?" THIS STRIP WOULD BE HIS, THIS STRIP WOULD BE HERS, AND THIS STRIP WOULD BE HERS.

"And how many boxes of candy would he get?" TWO AND A HALF. She said this for each share.

Following the three problem settings, Toe was questioned further. Pointing to a fourth of a box of candy, she was asked: "How much candy is that?" THIS IS THREE. "What part of the box is that?" A HALF. NO, A QUARTER. "What is it?" A QUARTER. "Show me half of the box." She did. "Show me a quarter of the box." She did. THERE. THAT'S A QUARTER, THAT'S A QUARTER, THAT'S A QUARTER, THAT'S A QUARTER. "If we had three children and one were to get that (a third of the box), how much of the box is that?" THREE PARTS. "Okay, could I call that half of the box?" YES. "Just this part?" YES. "Is that half of the box?" YES. "Is this (another third part) half of the box?" YES. She called each third, HALF of the box. "Could I call this (a third) a quarter of the box?" NO. "Why not?" BECAUSE IF IT WAS A QUARTER OF THE BOX, THERE WOULD BE ONE MORE STRIP OF CANDY. "What can I call all of that (a half)?" YOU COULD CALL THAT--A HALF YOU COULD CALL IT.